

GOODWILLIE CALCULI

BY

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THESIS

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Chapter 1

Introduction

The Goodwillie tower $\{P_n F\}$ of a homotopy functor F gives information about $F(X)$ only for X within the “radius of convergence”, where $F(X) = \text{holim } P_n F(X)$. On the boundary of the radius of convergence (where the connectivity of X is one below that required to guarantee convergence of the tower), one generally finds difficult problems. For instance, the Goodwillie tower of the identity functor from spaces to spaces (and other functors such as $Q(X)$ and $A(X)$) converges for simply connected spaces; questions on the ‘edge of the radius of convergence’ involve spaces whose first homotopy group is nontrivial. In general, if X is a nilpotent space, one may prove the same theorems as if $\pi_1 X = 0$, but little can be said if X is not nilpotent. This indicates that the maximum possible set of convergence for these functors should be “nilpotent spaces”, not “simply connected spaces”, the answer produced by ordinary Goodwillie calculus.

The Goodwillie tower is based on the idea of approximating a functor F by a series of functors $P_n F$ satisfying the very strong property of “ n -excision”. One might hope that by weakening this condition, one might obtain a larger radius of convergence. We begin our study by reviewing the weaker property of “ n -additive” and showing that for some functors, the weaker approximations give a larger radius of convergence.

These new constructions feature the left Kan extension in a prominent way. Briefly, given a full subcategory \mathcal{C} of the category of spaces \mathcal{T} , the left Kan extension $L_{\mathcal{C}}$ gives a way of constructing the adjoint to the restriction map for functors from \mathcal{T} . That is, there is an adjoint isomorphism:

$$\text{Hom}_{\mathcal{C}}(F, G|_{\mathcal{C}}) \cong \text{Hom}_{\mathcal{T}}(L_{\mathcal{C}} F, G).$$

The importance to us of the left Kan extension is that it defines a new functor $L_{\mathcal{C}} F(X)$ using only the behavior of F on \mathcal{C} and something about the relationship of \mathcal{C} to X . Specifically, $L_{\mathcal{C}} F(X)$ depends only on $F(\mathcal{C})$ and $\text{Map}(\mathcal{C}, X)$ for objects C of \mathcal{C} . In particular, no objects resembling $F(\text{Map}(\mathcal{C}, X))$ appear.

It turns out that the n -additive approximation of F can be expressed as Goodwillie’s n -excisive approximation applied to an associated functor $L^0 F_X$,

which is the left Kan extension (from finite sets = coproducts of S^0) of the functor $F_X(-) = F(X \wedge -)$. This suggests that the first thing one should investigate is the left Kan extensions of functors. We begin by investigating left Kan extensions of functors from spaces to spectra, since that case is generally much simpler to understand than the case of functors from spaces to spaces. In this case, the left Kan extensions along the full subcategory \mathcal{C}_n generated by $\{\bigvee^k S^0 \mid k \leq n\}$ turns out to classify all degree n functors from spaces to spectra, so we obtain a complete understanding of all functors of finite degree from spaces to spectra in this way.

Functors from spaces to spaces are much more complicated, but we can use our results on functors from spaces to spectra to understand the Goodwillie derivatives $D_n F$ of any functor, since these functors factor through the category of spectra as $D_n F(X) = \Omega^\infty(\mathbf{C}_n \wedge X^{\wedge n})$, for \mathbf{C}_n a spectrum. In particular, we show that if \mathbf{C}_n is connective, then $D_n F$ commutes with realizations. Using that result, we give a sufficient condition for an analytic functor to commute with realizations.

Once we understand functors from spaces to spectra as left Kan extensions, we can ask to what extent left Kan extensions of functors from spaces to spaces are interesting. The additive Goodwillie tower arises from $P_n(L^0 F_X)$, and the (ordinary) excisive Goodwillie tower arises from applying P_n to the left Kan extension L^∞ over all finite coproducts of spheres of the same dimension

$$\{\bigvee^k S^m \mid 0 \leq m \leq \infty \text{ and } k \geq 0\}.$$

Between $L^0 F$ and $L^\infty F$ lies an infinite sequence of Kan extensions $L^a F$, arising from using coproducts of spheres S^m , with $m \leq a$, equipped with natural transformations $L^a F \rightarrow L^{a+1} F$. This sequence could give rise to an entire family of “theories” $P_n^{(a)}$ between the additive and excisive approximations. The first step toward showing that this tower is interesting is to show that the approximations $P_n^{(a)} F$ can be distinct. We produce a family of examples, one for each a , such that $P_n^{(a)} F \not\cong P_n^{(a+1)} F$. We then go on to show that if F is an analytic functor, the tower

$$P_n(L^0 F) \rightarrow \cdots \rightarrow P_n(L^a F) \rightarrow \cdots \rightarrow P_n(L^\infty F)$$

stabilizes at a finite stage, so that Goodwillie’s $P_n F$ can actually be computed by examining a left Kan extension of finite dimension. This is interesting because the left Kan extension $L^a F$ requires “less” information than F to compute, since it depends only on the subcategory \mathcal{C} and maps from objects $C \in \mathcal{C}$ to X , and not arbitrarily high suspensions of X as $P_n F(X)$ requires. Fundamentally, even $P_n^{(0)}$ is an interesting functor, and understanding it is a necessary prerequisite to understanding the filtration of theories $P_n^{(a)}$. The n^{th} cross effect functor of F at X , denoted $\perp_n F(X)$, measures how much $F(\bigvee^n X)$ fails to be deter-

mined by the value of F on smaller coproducts of X . Since this is essentially exactly the “information” available to compute $P_n^{(0)}F(X)$, there should be a very close relationship between the two. One of the main results in this thesis is Theorem 9.1, which establishes that if F is reasonably good, there is a fibration sequence:

$$||\perp_{n+1}^{*+1} F(X)|| \rightarrow F(X) \rightarrow P_n^{(0)}F(X).$$

As a consequence of this theorem, we derive a spectral sequence with $E_{p,q}^1 = \pi_p \perp_{n+1}^q F(X)$ converging to $\pi_{p+q} P_n^{(0)}F(X)$. Also, this theorem gives us a way of relating the Goodwillie tower of the identity functor of (simplicial) groups to a derived functor of the lower central series.

This thesis is organized as follows. In Chapter 2, we review the basic categories and constructions used throughout. In Chapter 3, we give a brief exposition of some of Goodwillie’s calculus of functors, including n -cubes, the Blakers-Massey theorem, and some basic examples. In Chapter 4, we explain the left Kan extension and its homotopy invariant counterpart. In Chapter 5, we show that all degree n functors from spaces to spectra are left Kan extensions over the full subcategory of spaces containing $\bigvee^k S^0$, for $k = 0, \dots, n$. In Chapter 6, we show several results. Section 6.1 shows that analytic functors have connective coefficient spectra. Section 6.2 shows that Ω^∞ commutes with realizations of simplicial connective (*i.e.*, bounded at π_0) spectra. Section 6.3 combines these results to show that (reduced) analytic functors from spaces to spaces commute with realizations of simplicial k -connected spaces, where k is the larger of the radius of convergence or $-c$ for the universal analyticity constant c (see §3.6). Chapter 7 gives background on cotriples. Chapter 8 establishes basic properties of the $P_n^{(0)}$ and \perp_n constructions. In Chapter 9, we prove the main theorem (9.1), which establishes the relationship between the functor $P_n^{(0)}F$ and the $(n+1)^{\text{st}}$ cross effect. Chapter 10 elucidates some of the consequences of the main theorem, including the existence of a spectral sequence to calculate $P_n^{(0)}F(X)$ and the relationship with the work of Curtis on the lower central series of a simplicial group. Chapter 11 shows that there is a whole family of different theories interpolating between additive and excisive calculus, and all are distinct.

Chapter 2

Categories And Homotopy Invariance

There are two main categories we will study: pointed spaces and spectra. We will also be interested in simplicial objects in both of these categories. In this section, we explain exactly what we mean by these categories, and give a brief synopsis of the properties that we use.

2.1 Spaces

By “spaces” or “topological spaces”, we mean the topological category of compactly generated Hausdorff topological spaces with nondegenerate basepoint. In this category Hom is itself a topological space using the (compactly generated) compact-open topology. When we want to emphasize its nature as a space, we will write Map . Many convenient properties (such the continuity of the evaluation map from $X \times \text{Map}(X, Y)$ to Y) always hold in this category. The formation of various categorical constructions, such as product, requires a “compactification” of the topology on the product for arbitrary spaces. This is to be done implicitly wherever necessary. See [24] for more information about this. Henceforth, unless otherwise stated, the term “space” or “topological space” will mean an object of this category.

The pointed category has an object that is both initial and final; we use both $*$ and 0 to denote this object, depending on context.

2.2 Spectra

Spectra are the “stable category” associated to spaces. The so-called naïve spectra will be sufficient for our purposes. References for this material include Adams [1] and Kochman [19, Chapter 3.3], who also follows Adams’ treatment. In this category, a spectrum \mathbf{X} consists of a sequence of topological spaces $\{X_i \mid i \geq 0\}$ and structure maps $\Sigma X_i \rightarrow X_{i+1}$. By adjunction, the structure

maps may also be specified by a map $X_i \rightarrow \Omega X_{i+1}$; this is sometimes more convenient — see the example of \mathbf{HG} below. A morphism $f : \mathbf{X} \rightarrow \mathbf{Y}$ in this category is a (cofinal) sequence of maps $f_i : X_i \rightarrow Y_i$ that commute with the structure maps. (Cofinal means that the maps need not be defined for all i ; just on a cofinal subset of indices.) The homotopy groups here are $\pi_n \mathbf{X} = \operatorname{colim}_i \pi_{n+i} X_i$; a spectrum may have negative homotopy groups.

A nontrivial example of a spectrum is the Eilenberg-MacLane spectrum \mathbf{HG} . The spectrum \mathbf{HG} has $\mathbf{HG}_n = K(G, n)$, and the structure map is given by the canonical equivalence $K(G, n) \rightarrow \Omega K(G, n+1)$. Its only nonzero homotopy group is $\pi_0 = G$. This is an example of an “omega spectrum”: \mathbf{X} is called an omega spectrum if the adjoint structure map $X_n \rightarrow \Omega X_{n+1}$ is an equivalence. Every spectrum is equivalent to an omega spectrum. Another example is the sphere spectrum, frequently denoted \mathbf{S} or \mathbf{S}^0 , given by $(\mathbf{S})_n = S^n$, the n -sphere, and structure maps $\Sigma S^n \xrightarrow{\cong} S^{n+1}$. Its homotopy is stable homotopy, $\pi_n \mathbf{S} = \pi_n^S S^0$. This is a “suspension spectrum”—one in which the structure map $\Sigma X_n \rightarrow X_{n+1}$ is an isomorphism.

The homotopy category of spectra is a triangulated category, much like the homotopy category of chain complexes. Just as in the case of chain complexes, it is sometimes desirable to distinguish between arbitrary spectra and “bounded below” spectra, whose homotopy π_n vanishes for all $n \leq N$. The main important trait of bounded below spectra is that suspension increases their connectivity. We use the word “*connective*” to mean a spectrum that has no negative homotopy groups. In the literature, the word connective sometimes means bounded below.

The categories of spectra and spaces are related by a pair of adjoint functors, Ω^∞ and Σ^∞ . The functor $\Sigma^\infty : \mathbf{Spaces} \rightarrow \mathbf{Spectra}$ creates a spectrum from a space X by putting $(\Sigma^\infty X)_n = \Sigma^n X$, with the structure maps $\Sigma(\Sigma^\infty X)_n \xrightarrow{\cong} (\Sigma^\infty X)_{n+1}$. The functor $\Omega^\infty : \mathbf{Spectra} \rightarrow \mathbf{Spaces}$ sends \mathbf{X} to $\operatorname{colim}_n \Omega^n X_n$. Note that $\pi_n(\Omega^\infty \mathbf{X}) = \pi_n \mathbf{X}$ for all $n \geq 0$. In particular, π_0 and π_1 of an “infinite loop space” are abelian groups. The adjunction that arises from the familiar suspension-loop adjunction is

$$\operatorname{Hom}_{\mathbf{Spectra}}(\Sigma^\infty X, \mathbf{Y}) \cong \operatorname{Hom}_{\mathbf{Spaces}}(X, \Omega^\infty \mathbf{Y}).$$

When working with unbased spaces, the appropriate functor to use is $\Sigma_+^\infty(X)$, which is the suspension spectrum of X taken after a disjoint basepoint is added.

The category of spectra has the very useful property that fibration sequences and cofibration sequences are equivalent. The proof of this uses the Blakers-Massey theorem (3.4), so it appears later, as Corollary 3.8 on page 19. The equivalence of fibration and cofibration sequences implies that the fiber of a map is naturally equivalent to the loop spectrum of the cofiber of the map.

When we have groups acting on spectra, we will always be in a situation where it is appropriate to use naïve G -spectra. These are spectra \mathbf{X} in which G

acts on each X_n , and the structure maps $S^1 \wedge \mathbf{X}_n \rightarrow \mathbf{X}_{n+1}$ are G -equivariant, with G acting trivially on the suspension coordinate. The simplest G -spectra are those which are suspension spectra of spaces with a free G -action.

2.3 Functors

The category of topological spaces (or spectra) is enriched over topological spaces (respectively, spectra), meaning that the Hom sets can be given the structure of a topological space (respectively, spectrum). We require that our functors respect this additional structure.

Let \mathcal{C} and \mathcal{D} be topological categories. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. In the standard terminology, F is called *continuous* if the map $f \mapsto F(f)$ induces a continuous map $\mathrm{Hom}_{\mathcal{C}}(A, B) \rightarrow \mathrm{Hom}_{\mathcal{D}}(FA, FB)$.

We require that all functors be continuous.

2.4 Simplicial Objects

We will also be interested in the simplicial objects in the categories of spaces and spectra: simplicial spaces and simplicial spectra. The standard reference for all facts about simplicial objects is May’s book [21], but Curtis’s award-winning exposition [12] is a more accessible place to start. Weibel [26, Chapter 8] is a concise but valuable reference. The recent publication by Goerss and Jardine [15] is another resource for facts about simplicial homotopy theory. The reader completely unfamiliar with the subject is advised to consult one of these references; this is just a very brief review of some relevant facts.

Before delving into the definitions (which are notoriously opaque), let us consider why simplicial objects are so important. In general, simplicial objects add another “dimension” to a category; for instance, simplicial abelian groups are equivalent to chain complexes (bounded ≥ 0) of abelian groups (this is known as the Dold-Kan correspondence). Adding this dimension provides a setting for homological algebra by providing a category in which projective resolutions of abelian groups can live. In this case, there is no way to “reduce” a projective resolution back down to an ordinary abelian group without losing the information it provides. In the case of simplicial spaces, however, the base category (spaces) already has enough structure that it is possible to reduce a simplicial space back down to an ordinary space without losing information.¹ This process is called “realization”, and plays a central role in the work in this thesis.

Let Δ denote the category of ordered finite sets whose objects are $\{[n] \mid n \geq 0\}$, with $[n] = \{0 < \dots < n\}$, and whose morphisms are nondecreasing set maps. A

¹This actually might be said to occur because spaces are equivalent to simplicial sets, so they already “contain one simplicial dimension”, and the Eilenberg-Zilber theorem (2.8) shows that nothing more is gained by adding more simplicial dimensions.

simplicial object in any given category is a functor from the opposite category of Δ , denoted Δ^{op} , to the given category. The behavior of a functor $\Delta^{\text{op}} \rightarrow \mathcal{C}$ is determined by its values on the objects and on certain morphisms called “face” and “degeneracy” maps. In Δ , there are $n + 1$ face maps $\delta_i : [n] \rightarrow [n - 1]$:

$$\delta_i(j) = \begin{cases} j & \text{if } j \leq i \\ j - 1 & \text{if } j > i \end{cases},$$

and $n + 1$ degeneracy maps $\sigma_i : [n] \rightarrow [n + 1]$:

$$\sigma_i(j) = \begin{cases} j & \text{if } j < i \\ j + 1 & \text{if } j \geq i \end{cases}.$$

Generally a simplicial object is denoted by X_\bullet , and its value on objects is $X_n = X([n])$. The image of the face maps are the $d_i = X(\delta_i)$, and the image of the degeneracy maps are the $s_j = X(\sigma_j)$.

Let Δ^n denote the standard n -simplex. The realization $\|X_\bullet\|$ of a simplicial space X_\bullet is taken to be the colimit of the following process. Let $R_0 = X_0$, and proceed by induction to let R_n be the pushout of the following diagram:

$$R_{n-1} \longleftarrow X_n \times \partial\Delta^n \longrightarrow X_n \times \Delta^n,$$

where the left map is given by $(x, p) \mapsto (d_i x)$ when p is an element of the i -th face of Δ^n . The definition of realization given here is called the “fat” realization; when we need to refer to the usual definition (which uses the quotient of X_n by the degeneracies where we have used X_n), we will say “strict realization” and denote it $|X_\bullet|$. As discussed in Section 2.7 below, if the degeneracy maps $s_j : X_{n-1} \rightarrow X_n$ are not cofibrations, the quotient may not be a “homotopy invariant”. However, in the case of simplicial sets, all injections are cofibrations, so there is never an issue when working with simplicial sets.

One very important fact about realization is that it is homotopy invariant in the following sense.

Lemma 2.1 (Realization Lemma). *([23, Proposition A.1, p. 308]) Let X_\bullet and Y_\bullet be simplicial spaces, and suppose $f : X_\bullet \rightarrow Y_\bullet$ is a simplicial map with f_n a weak equivalence for all n . Then $\|f\|$ is a weak equivalence. \square*

We will sometimes want to use the strict realization and know that it is a homotopy invariant. This happens if the simplicial space being realized is “good”.

Definition 2.2 (Good simplicial space). A simplicial space is called *good* if all of the degeneracy maps $s_j : X_n \rightarrow X_{n+1}$ are closed cofibrations.

If a simplicial space is good, then both the “fat” and “strict” realizations are equivalent.

Theorem 2.3. ([23, Proposition A.2, p. 308]) *If X is a good simplicial space, then the natural map $\|X\| \rightarrow |X|$ is a (weak homotopy) equivalence. \square*

Corollary 2.4. *Let X_{\bullet} be a bisimplicial space. If each simplicial space $[j] \mapsto X_{i,j}$ is good and $[i] \mapsto X_{i,j}$ is good, then the natural map $\|X\| \rightarrow |X|$ between the realizations in one direction and another is a weak homotopy equivalence. For this reason, we call such bisimplicial spaces “good” as well.*

Proof. Since the realization of levelwise cofibrations is a cofibration, it suffices to show that the degeneracy maps in each simplicial direction of a multi-simplicial space are cofibrations. (Our spaces are all Hausdorff by hypothesis, so cofibration implies closed.) Taking the realizations in one direction at a time, this follows from Theorem 2.3. \square

Lemma 2.5. *Many operations preserve “goodness”. Let X be a good simplicial space (where each space has a nondegenerate basepoint), and let Z be a space (with a nondegenerate basepoint). Then the following simplicial spaces are good:*

1. $(X)_+$ (even if X does not have a nondegenerate basepoint)
2. $Z \vee X$.
3. $Z \times X$.
4. $Z \wedge X$.
5. $\text{Map}(C, X)$, for any compact cofibrant C

If each $X(k)$ is a good simplicial space, then:

6. $\text{diag} \left(\bigvee_{i=1}^k X(i) \right)$ is good

And finally, if $[i, j] \mapsto X(i)_j$ is a bisimplicial space with each $X(i)$ good, then:

7. the realization $\|[i] \mapsto X(i)\|$ is a good simplicial space

Proof. We only need to make these arguments in the category of spaces, so when it is convenient, we can use a characterization of cofibrations that is specific to that category.

Item 1: obvious. Item 2: coproduct (colimit) of cofibrations is a cofibration. Item 3: follows from characterization of cofibrations via neighborhood deformation retracts (as in [10, Theorem VII.1.5, p. 431]). Item 4: the map in question is the pushout (colimit) of vertical cofibrations in the following diagram:

$$\begin{array}{ccccc}
 * & \xleftarrow{\quad} & Z \vee X_n & \xrightarrow{\quad} & Z \times X_n \\
 \downarrow & & \downarrow & & \downarrow \\
 * & \xleftarrow{\quad} & Z \vee X_{n+1} & \xrightarrow{\quad} & Z \times X_{n+1}
 \end{array}$$

Item 5: the neighborhood retraction for X_n in X_{n+1} induces a neighborhood retraction of $\text{Map}(C, X_n)$ in $\text{Map}(C, X_{n+1})$ using the height function $\phi(f) =$

$\sup_{c \in C} \phi_X(f(c))$ derived from the height function ϕ_X for s_n . Item 6: For a finite coproduct, this follows from (2) along with the fact that the composition of cofibrations is a cofibration (the diagonal degeneracies come from composing the degeneracies in each individual direction). Item 7: Each map $s_{i,j} : X(i)_j \rightarrow X(i)_{j+1}$ is a cofibration, so the realization in the i direction preserves the cofibrations, producing a cofibration $\|X(\cdot)_j\| \rightarrow \|X(\cdot)_{j+1}\|$. \square

In this paper we work with the category of topological spaces because homotopy inverse limit constructions are very important, and these require fibrant objects to be well-behaved. When working with simplicial sets, it is more effort to maintain fibrancy. However, some standard results that we use are proven for bisimplicial spaces, so we need to establish that they also hold for topological spaces.

To this end, we recall some facts about simplicial and bisimplicial sets. Given a simplicial space X , the singular set functor, $\text{Sing}(X)$ produces a simplicial set whose k -simplices are the set (not space) of continuous maps of the standard topological k -simplex into X ; that is, $\text{Hom}(\Delta^k, X)$. The functor Sing is right adjoint to the strict realization functor, and the map $|\text{Sing}(X)| \rightarrow X$ is always a weak equivalence. These facts and more can be found in [15, Chapter 1]. Given a bisimplicial set X , there is a functor “Tot” that produces a simplicial set. Let $\Delta[m]$ be the standard simplicial n -simplex, $\text{Hom}_\Delta(-, [m])$, let Δ^m be standard m -simplex that is the strict realization of $\Delta[m]$, and let $X_{m,*}$ denote the simplicial set $[k] \mapsto X_{m,k}$. This functor “Tot” can be described as the coequalizer of the diagram:

$$\bigsqcup_{\alpha:[m] \rightarrow [n]} X_{m,*} \times \Delta[n] \rightrightarrows \bigsqcup_{[m]} X_{m,*} \times \Delta[m],$$

where the first coproduct is taken over all morphisms in Δ , and the second is taken over all objects in Δ . The first morphism sends (x, y) to (α^*x, y) and the second morphism sends (x, y) to (x, α_*y) .

Applying the *strict* geometric realization functor (which commutes with coproducts and finite products) to this diagram produces a diagram

$$\bigsqcup_{\alpha:[m] \rightarrow [n]} |X_{m,*}| \times \Delta^n \rightrightarrows \bigsqcup_{[m]} |X_{m,*}| \times \Delta^m.$$

The coequalizer of this diagram is the realization $\|[i] \mapsto [j] \mapsto X_{i,j}\|$. But strict realization is a left adjoint, and hence preserves coequalizers, so we have established:

Lemma 2.6. *Let X be a bisimplicial set. Then using strict realizations, $|\text{Tot}(X..)|$ is isomorphic (homeomorphic) to $\|[i] \mapsto [j] \mapsto X_{i,j}\|$. \square*

Corollary 2.7. *Let $X..$ be a simplicial space and $\text{Sing } X..$ be the bisimplicial set*

formed by applying the singularization functor to each X_i . Then we have:

$$||\text{Tot}(\text{Sing. } X.)|| \simeq ||X.||.$$

Proof. Let Y be the simplicial set $Y_{i,j} = \text{Sing}_j X_i$. Since $\text{Tot}(Y)$ is a simplicial set, the (fat) realization and strict realizations are equivalent, so we can work with the strict realization. Lemma 2.6 then gives:

$$|\text{Tot}(Y.)| \cong |[i] \mapsto |[j] \mapsto Y_{i,j}|.$$

The inner realization on the right is $|[j] \mapsto Y_{i,j}| = |\text{Sing}(X_i)| \simeq X_i$. The functor Sing takes inclusions to cofibrations, so the simplicial space $|\text{Sing}(X_i)|$ is good; hence the strict realization that appears here is equivalent to the fat realization. We can then use the fact that the fat realization is a homotopy functor, so the weak equivalences $|\text{Sing}(X_i)| \simeq X_i$ induce an equivalence of (fat) realizations:

$$|[i] \mapsto |\text{Sing}(X_i)|| \xrightarrow{\simeq} |[i] \mapsto X_i|$$

Chaining the equivalences together produces the desired result. \square

The degeneracy maps encode “redundant” information that is necessary for the proper homotopical behavior of the object. One important consequence of the existence of the degeneracy maps is the Eilenberg-Zilber theorem. The Eilenberg-Zilber theorem for bisimplicial sets relates the Tot of a bisimplicial space to its diagonal. The diagonal of a bisimplicial object is $\text{diag}(X..) = X_{n,n}$.

Theorem 2.8 (Eilenberg-Zilber). *([8, Proposition B.1, p. 119]) Let X be a bisimplicial set. There is a natural isomorphism of simplicial sets $\text{Tot}(X) \xrightarrow{\cong} \text{diag}(X)$. \square*

We actually want to use the following statement for bisimplicial spaces:

Corollary 2.9. *Let $X..$ be a good bisimplicial space. The realization in one direction and then the other, $|X..|$, is naturally homotopy equivalent to $||\text{diag } X..||$. Realization is a homotopy colimit, and homotopy colimits commute up to natural isomorphism, so the order in which the realizations are taken does not matter.*

Proof. Since X is good, the (fat) realization $||X..||$ in the statement of the theorem is equivalent to the strict realization. Using strict realizations, we have a homeomorphism:

$$|\text{diag}(X)| \xrightarrow{\cong} |\text{Tot}(X)| \cong |[i] \mapsto |[j] \mapsto X_{i,j}|.$$

The construction of Tot for bisimplicial sets prior to Lemma 2.6, and the maps in the Eilenberg-Zilber theorem, have direct translations to bisimplicial spaces once we use the strict realization. This translation gives the equivalence above; it remains to check that $\text{diag}(X)$ is a good space, so that its strict realization

agrees with the (fat) realization. The degeneracies in $\text{diag}(X)$ are compositions of “horizontal” and “vertical” degeneracies, both of which are cofibrations by our hypothesis that X is a good bisimplicial space, so $\text{diag}(X)$ is a good simplicial space. \square

The realization of a levelwise fibration of simplicial spaces need not be a fibration, but with some conditions it is. The following lemma is stated in [25] for bisimplicial sets; we will not argue that it is also true for simplicial spaces, since it is also an easy corollary of Theorem 2.12, below.

Lemma 2.10. *([25, Lemma 5.2, p. 165]) Let $X. \rightarrow Y. \rightarrow Z.$ be map of simplicial spaces such that each $X_n \rightarrow Y_n \rightarrow Z_n$ is a fibration up to homotopy. If each Z_n is connected, then $\|X.\| \rightarrow \|Y.\| \rightarrow \|Z.\|$ is a fibration up to homotopy. \square*

We use a generalization of this result to 2-cubes, due to Bousfield and Friedlander, heavily later in this work. They define a fibrancy condition called the π_* -Kan condition.

Definition 2.11. A simplicial space $X.$ is said to satisfy the π_* -Kan condition if:

- for any $m \geq 1$ and any $t \geq 1$, and for any point $a \in X_m$, any coherent collection (in the sense of the usual fibrancy condition: $\partial_i x_j = \partial_{j-1} x_i$ for $i > j$ with $i, j \neq k$) of elements $x_i \in \pi_t(X_{m-1}, \partial_i a)$ (for $0 \leq i \leq m$, and $i \neq k$), there exists a $y \in \pi_t(X_m, a)$ with $\partial_i y = x_i$; and
- the simplicial set $\pi_0(X.)$ is fibrant.

For instance, a simplicial space $X.$ certainly satisfies the π_* -Kan condition if each X_i is connected. Also, simplicial spaces arising from bisimplicial groups satisfy the π_* -Kan condition.

Theorem 2.12 (Bousfield-Friedlander). *([8, Theorem B.4, p. 121]) Let*

$$\begin{array}{ccc} V. & \longrightarrow & X. \\ \downarrow & & \downarrow \\ W. & \longrightarrow & Y. \end{array}$$

be a commutative square of simplicial spaces such that for each n , the square consisting of V_n , W_n , X_n , and Y_n is a homotopy pullback square. If X and Y satisfy the π_ -Kan condition and if $\pi_0 X. \rightarrow \pi_0 Y.$ is a fibration of simplicial sets, then after realization we have a homotopy pullback square:*

$$\begin{array}{ccc} \|V.\| & \longrightarrow & \|X.\| \\ \downarrow & & \downarrow \\ \|W.\| & \longrightarrow & \|Y.\| \end{array}$$

Bousfield and Friedlander actually prove their theorem for bisimplicial sets, so some comments are in order to apply it to simplicial spaces. The functor Sing is a right adjoint, so it preserves inverse limits; in particular, if \mathcal{X} is a homotopy pullback cube of spaces, then $\text{Sing } \mathcal{X}$ is a homotopy pullback cube of simplicial sets. The bisimplicial set $\text{Sing } Y$ satisfies the π_* -Kan condition if and only if the simplicial space Y does, since it is a condition on homotopy groups, and the homotopy groups of $\text{Sing } Y$ and Y are isomorphic for any space Y . Now starting with a commutative square of simplicial spaces satisfying the hypotheses stated above, we apply Sing to produce a commutative square of bisimplicial sets satisfying the analogous hypotheses used by Bousfield and Friedlander. Their result is then that the square of simplicial sets formed by taking the diagonal is Cartesian. Then $\text{diag}(\text{Sing } X) \cong \text{Tot}(\text{Sing } X)$ and $\|\text{Tot}(\text{Sing } X)\| \simeq \|X\|$ (Corollary 2.7), producing the result as we state it. \square

The realization of a simplicial spectrum requires that we define both the spaces in the realization and structure maps. Begin with a simplicial spectrum $[m] \mapsto \mathbf{X}_m$ with the spectrum \mathbf{X}_m consisting of spaces $X_{m,n}$ and structure maps $S^1 \wedge X_{m,n} \rightarrow X_{m,n+1}$. Define the realization of this simplicial spectrum $\|[m] \mapsto \mathbf{X}_m\|$ to have n^{th} space $\|[m] \mapsto X_{m,n}\|$. Recall that the suspension of X is homeomorphic to $\text{hocolim}(* \leftarrow X \rightarrow *)$ in the category of pointed spaces. The structure maps are given by commuting the realization (which is a homotopy colimit) with the suspension (which is also a homotopy colimit) and using the structure map of each \mathbf{X}_m in the following manner:

$$\begin{aligned} S^1 \wedge \|[m] \mapsto \mathbf{X}_m\|_n &= S^1 \wedge \|[m] \mapsto X_{m,n}\| \\ &\simeq \|[m] \mapsto S^1 \wedge X_{m,n}\| \\ &\rightarrow \|[m] \mapsto X_{m,n+1}\| \\ &= \|[m] \mapsto \mathbf{X}_m\|_{n+1} \end{aligned}$$

2.5 The Nerve Of A Category

A category \mathcal{C} determines a simplicial set called the nerve of \mathcal{C} , denoted $N\mathcal{C}$. The n -simplices of this object consist of n composable morphisms in the category; for $n = 0$, we define $N_0\mathcal{C} = \text{Obj}(\mathcal{C})$ (or alternatively, consider only the identity morphisms). The face maps are given by composing two adjacent morphisms, or deleting them at the extrema, and the degeneracy maps are given by inserting identity morphisms. Explicitly, let $\alpha \in N_n\mathcal{C}$ be a sequence of n composable morphisms:

$$\alpha = (C_n \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_0} C_0).$$

Then the faces of α are given by:

$$d_i \alpha = \begin{cases} C_n \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_1} C_1 & \text{if } i = 0 \\ C_n \xrightarrow{\alpha_{n-1}} \cdots C_{i+1} \xrightarrow{\alpha_{i-1}\alpha_i} C_{i-1} \rightarrow \cdots \xrightarrow{\alpha_0} C_0 & \text{if } 0 < i < n \\ C_{n-1} \xrightarrow{\alpha_{n-2}} \cdots \xrightarrow{\alpha_0} C_0 & \text{if } i = n \end{cases}$$

The degeneracies of α are given by:

$$s_j \alpha = \left\{ C_n \rightarrow \cdots \rightarrow C_j \xrightarrow{\alpha_j} C_j \rightarrow \cdots \rightarrow C_0 \quad \text{for } 0 \leq i \leq n \right.$$

For example, the category

$$C_2 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_0} C_0$$

has as its nerve the following simplicial object:

- in dimension zero: three nondegenerate simplices, C_0 , C_1 , and C_2 ;
- in dimension one: three nondegenerate simplices:

$$\begin{aligned} C_1 &\xrightarrow{\alpha_0} C_0 \\ C_2 &\xrightarrow{\alpha_1} C_1 \\ C_2 &\xrightarrow{\alpha_0\alpha_1} C_0, \end{aligned}$$

plus three more (degenerate) simplices that correspond to the identity maps of C_0 , C_1 , and C_2 ;

- in dimension two: one nondegenerate simplex: $C_2 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_0} C_0$, and six degenerate simplices: $C_2 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_0} C_1$, *etc.*
- in higher dimensions: degenerate simplices only.

To illustrate the action of the face maps, consider their action on the 2-simplex $C_2 \rightarrow C_1 \rightarrow C_0$:

$$\begin{aligned} d_0(C_2 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_0} C_0) &= C_2 \xrightarrow{\alpha_1} C_1 \\ d_1(C_2 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_0} C_0) &= C_2 \xrightarrow{\alpha_0\alpha_1} C_0 \\ d_2(C_2 \xrightarrow{\alpha_1} C_1 \xrightarrow{\alpha_0} C_0) &= C_1 \xrightarrow{\alpha_0} C_0 \end{aligned}$$

2.6 Equivalences And Connectivity

In any of these categories, a map is k -connected if it is an isomorphism on π_j for $j < k$ and surjective on π_k . An object is k -connected if the map *from* the initial object is k -connected. Note that this means that S^n is an $(n-1)$ -connected space. A spectrum is called connective if all of its negative homotopy groups are zero.

A map of spectra is an equivalence if it is an isomorphism on π_* . A map of spaces is an equivalence if it induces a bijection on π_0 and an isomorphism on π_* for all compatible choices of basepoint (not just the basepoint with which all pointed spaces are equipped).

2.7 Homotopy Invariance

The basic object of study of diverse variations of Goodwillie calculus is a homotopy functor. A homotopy functor is a functor that preserves equivalences. In our setting, we will mainly consider functors from pointed spaces to pointed spaces. It turns out that the study of these functors is intimately tied up with the study of functors from pointed spaces to spectra, so we will also be interested in those. For various examples, it is more convenient to consider algebraic settings, such as functors from spaces to chain complexes of abelian groups. Generally these embed into the category of spectra or spaces in some manner that should be clear upon reflection. For instance, integral homology $H_*(X; \mathbb{Z})$ is generally regarded as the homology of a chain complex of abelian groups, but is also $\pi_*(\mathbf{H}\mathbb{Z} \wedge X)$ or $\pi_*\Omega^\infty(\mathbf{H}\mathbb{Z} \wedge X)$, which provides a sensible way of considering homology as the homotopy of a spectrum or space.

Although homotopy functors, such as π_* itself, homology, and loops on a space, are abundant, there are many familiar functors that are *not* homotopy functors. For example, the pushout is not a homotopy functor because the diagram

$$* \longleftarrow S^0 \longrightarrow *$$

has as its pushout one point, but there is an equivalence of diagrams (an honest map of diagrams that is an equivalence on each vertex) between this one and

$$D^1 \longleftarrow S^0 \longrightarrow D^1,$$

whose pushout is S^1 . This is just the beginning of trouble; there are simplicial spaces whose strict realization is not equivalent to the “fat” realization used in this paper.

Since we are interested in studying only homotopy functors, and we do not want to be constantly concerned whether various constructions are homotopy invariant, we make the blanket assertion that *all* constructions will be made in a homotopy invariant way. In particular, all colimits will be homotopy colimits, and all inverse limits will be homotopy inverse limits. In order to remind the reader, we will use the symbols $\mathrm{hocolim}$ and holim for these constructions. We will point out explicitly other places where homotopy-invariant constructions are necessary as they arise. Three situations are worth mentioning in particular:

- If X and Y have nondegenerate basepoints, then the standard coproduct $X \vee Y$ is a homotopy invariant, so there is no need to think of a special

coproduct occurring.

- If X and Y are spaces, then $X \times Y$ is a homotopy invariant.
- If X and Y are spaces with nondegenerate basepoints, then $X \wedge Y$ is homotopy invariant. In this case the inclusion $X \vee Y \rightarrow X \times Y$ is a cofibration, so the strict cofiber ($= X \wedge Y$) is a homotopy invariant.
- If X is a CW complex and Y is any space, then $\text{Map}(X, Y)$ is a homotopy invariant construction, so there is no need to take a special Map as long as the source is a CW complex.

In his book *Homotopical Algebra* [22], Quillen developed a general framework for understanding problems of homotopy invariance, called “model categories”. Quillen’s work codifies the general understanding that one should make sure that colimit constructions involve cofibrations (“cofibrant objects”) and inverse limit constructions involve fibrations (“fibrant objects”). Dwyer and Spalinski [13] have written an excellent introduction to Quillen’s work, full of examples familiar to the working topologist or algebraist. In the category of topological spaces with the model structure where weak equivalences are π_* -isomorphisms and fibrations are Serre fibrations, all objects are fibrant, and CW complexes are cofibrant. In the category of simplicial sets with π_* -isomorphisms for weak equivalences and Kan fibrations for fibrations, all objects are cofibrant, but only Kan complexes are fibrant.

Chapter 3

Goodwillie Calculus

3.1 n -cubes

We must first lay out the notation and vocabulary we will use. Given a set T , define the category $\mathcal{P}(T)$ to have objects all subsets of T and morphisms the inclusions of subsets. A T -cube \mathcal{X} is a functor defined on $\mathcal{P}(T)$. In general, the functor \mathcal{X} will take values in the category of spaces or spectra. An n -cube is a T -cube with $|T| = n$. When there is only one n -cube being discussed, we may let \mathbf{n} denote the set $\{1, \dots, n\}$, and speak simply of an \mathbf{n} -cube. A $\mathbf{2}$ -cube \mathcal{X} is a diagram like this:

$$\begin{array}{ccc} \mathcal{X}(\emptyset) & \longrightarrow & \mathcal{X}(\{1\}) \\ \downarrow & & \downarrow \\ \mathcal{X}(\{2\}) & \longrightarrow & \mathcal{X}(\{1, 2\}) \end{array}$$

The “initial” object in the cube is $\mathcal{X}(\emptyset)$ and the “terminal” object is $\mathcal{X}(\{1, 2\}) = \mathcal{X}(\mathbf{2})$. We will frequently refer to those particular two objects in any cube. When we want to consider the relationship between the initial object and the rest of the cube, we will use the category $\mathcal{P}_0(\mathbf{n}) = \mathcal{P}(\mathbf{n}) - \{\emptyset\}$, and use the inverse limit over this category to assemble the information about all of the objects except $\mathcal{X}(\emptyset)$. Similarly, if we want to consider the relationship between the final object and the rest of the cube, we will use the category $\mathcal{P}_1(\mathbf{n}) = \mathcal{P}(\mathbf{n}) - \{\mathbf{n}\}$.

Definition 3.1 (Cartesian). An \mathbf{n} -cube \mathcal{X} is “Cartesian” if the map

$$\mathcal{X}(\emptyset) \rightarrow \operatorname{holim}_{U \in \mathcal{P}_0(\mathbf{n})} \mathcal{X}(U)$$

is an equivalence.

Alternatively, \mathcal{X} is Cartesian if it is a homotopy pullback cube; that is, $\mathcal{X}(\emptyset)$ is equivalent to the homotopy inverse limit of the rest of the cube. A 2-cube that is a pullback cube is guaranteed to be a homotopy pullback if one of the maps to the terminal object is a fibration.

Definition 3.2 (co-Cartesian). An \mathbf{n} -cube \mathcal{X} is “co-Cartesian” if the map

$$\operatorname{hocolim}_{U \in \mathcal{P}_1(\mathbf{n})} \mathcal{X}(U) \rightarrow \mathcal{X}(\mathbf{n})$$

is an equivalence.

Alternatively, the cube is co-Cartesian if it is a homotopy pushout. \mathcal{X} is said to be strongly co-Cartesian if every two dimensional sub-cube is co-Cartesian. A 2-cube that is a pushout cube is guaranteed to be a homotopy pushout if one of the maps from the initial object is a cofibration.

A cube \mathcal{X} is strongly co-Cartesian if every 2-cube contained in \mathcal{X} is co-Cartesian. A cube is said to be k -Cartesian if the map $\mathcal{X}(\emptyset) \rightarrow \operatorname{holim}_{U \in \mathcal{P}_0(\mathbf{n})} \mathcal{X}(U)$ is k -connected.

The following cube, which forms the suspension of X , is an example of a co-Cartesian 2-cube.

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

By the Freudenthal suspension theorem, we see that when $X = S^n$, this cube is also $(2n - 1)$ -Cartesian.

Theorem 3.3 (Freudenthal). *For $n \geq 1$, the map $S^n \rightarrow \Omega S^{n+1}$ is $(2n - 1)$ -connected. \square*

3.2 The Blakers-Massey Theorem And Its Consequences

The Blakers-Massey theorem is closely related to the Freudenthal suspension theorem and the Eilenberg-Zilber theorem for bisimplicial sets. It gives a way to understand the homotopy of a pushout in a range, in terms of the spaces used to construct it. The most often-used statement of the theorem, as proven by Ellis and Steiner, follows.

Theorem 3.4 (Ellis-Steiner). *[3, 14] Let \mathcal{X} be a strongly co-Cartesian \mathbf{n} -cube of spaces ($n \geq 1$), with each map $\mathcal{X}(\emptyset) \rightarrow \mathcal{X}(\{i\})$ being k_i -connected. Then \mathcal{X} is $((1 - n) + \sum k_i)$ Cartesian. \square*

In particular, this immediately implies the Freudenthal theorem since the map from S^n to the cone on S^n is n -connected, so the 2-cube computing ΣS^n is $(1 - 2) + (n + n) = 2n - 1$ connected.

In at least one delicate calculation, we will have occasion to use the full strength of the theorem that Goodwillie proves.

Theorem 3.5 (Goodwillie). *Let \mathcal{X} be an S -cube, with $|S| \geq 1$. Suppose that*

1. for each nonempty $T \subset S$, the sub- T -cube of \mathcal{X} induced by the inclusion of T into S is $k(T)$ -co-Cartesian, and
2. $k(U) \leq k(T)$ whenever $U \subset T$.

Then \mathcal{X} is k -Cartesian, where k is the minimum of $(1 - |S|) + \sum_{\alpha} k(T_{\alpha})$ over all partitions $\{T_{\alpha}\}$ of S by nonempty sets. \square

Using the Blakers-Massey theorem, we can now prove some important properties of spectra. We will begin by proving the basic property of spectra that Ω and Σ are inverse operations on the homotopy category. This is what is meant by spectra being a “stable” category.

Let \mathbf{X} be a spectrum. Since \mathbf{X} is equivalent to an omega spectrum, and we are only interested to behavior up to homotopy, we may assume that \mathbf{X} is an omega spectrum.

First suppose \mathbf{X} is a bounded below omega spectrum (so π_j is zero for all $j \ll 0$). \mathbf{X} is an omega spectrum, so $\pi_j X_n = \pi_{j+1} X_{n+1} = \cdots = \pi_{(j-n)} \mathbf{X}$. Since \mathbf{X} is bounded below, $\pi_{(j-n)} \mathbf{X} = 0$ for $j - n \ll 0$, which shows that there is a (not necessarily positive) constant c such that $\pi_j X_n = 0$ for $j < n + c$. That is, X_n is roughly $(n + c)$ -connected.

We will use this fact and the Blakers-Massey theorem to prove that $\mathbf{X} \rightarrow \Omega \Sigma \mathbf{X}$ is an equivalence by showing that the map is at least m -connected for arbitrary m . Let $N(m)$ be large enough that X_n is m -connected for all $n \geq N(m)$. By Theorem 3.4, the map $X_n \rightarrow \Omega \Sigma X_n$ is $(2m - 1)$ -connected, which is certainly more than m -connected. This holds for all $n \geq N(m)$, and the homotopy type of a spectrum only depends on a cofinal subset of the spaces that compose it, so this shows that $\mathbf{X} \rightarrow \Omega \Sigma \mathbf{X}$ is at least m -connected, with m arbitrary. Therefore, the map must be an equivalence. If \mathbf{X} is not bounded below, then write \mathbf{X} as the homotopy colimit of bounded below spectra $\mathbf{X}\langle m \rangle$ created by taking the $(m + n)$ -th connective cover $X_n\langle m + n \rangle$ of each X_n . (The spectrum $\mathbf{X}\langle m \rangle$ has $\pi_j = 0$ for $j < m$.) Since the m -th connective cover of \mathbf{X} comes equipped with a map to \mathbf{X} , this gives us a sequence $\mathbf{X} = \text{hocolim}_{m \rightarrow -\infty} \mathbf{X}\langle m \rangle$. From the bounded below case, we have an equivalence on each object $\mathbf{X}\langle m \rangle \rightarrow \Omega \Sigma(\mathbf{X}\langle m \rangle)$. The homotopy colimit of a map of diagrams that is a weak equivalence on each object is itself a weak equivalence; this provides the required equivalence $\mathbf{X} \simeq \text{hocolim}_{m \rightarrow -\infty} \mathbf{X}\langle m \rangle \simeq \text{hocolim}_{m \rightarrow -\infty} \Omega \Sigma(\mathbf{X}\langle m \rangle) \simeq \Omega \Sigma \mathbf{X}$.

A similar argument using the dual Blakers-Massey theorem shows that $\Sigma \Omega \mathbf{X} \rightarrow \mathbf{X}$ is an equivalence as well. This establishes:

Lemma 3.6. *In the homotopy category of spectra, Ω and Σ are inverse operations.*

A very similar argument to the one in Lemma 3.6 shows the following:

Lemma 3.7. *If \mathcal{X} is a co-Cartesian cube of spectra, then \mathcal{X} is also a Cartesian cube.*

In particular, this implies:

Corollary 3.8. *If $X \rightarrow Y \rightarrow Z$ is a cofibration sequence of spectra, then it is equivalent to a fibration sequence.*

Proof. A cofibration sequence is (up to homotopy), a pushout cube

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ CX & \longrightarrow & Z \end{array},$$

so the previous lemma applies. \square

As a functional corollary of the fact that Ω and Σ are inverse operations, we find that Σ commutes with homotopy inverse limits, and Ω commutes with homotopy colimits.

Corollary 3.9. *In the homotopy category of spectra, Ω commutes with hocolim , and Σ commutes with holim .*

Proof. Let X be a functor from an unspecified diagram category to spectra. We have $\Omega \text{holim } \Sigma X \simeq \text{holim } \Omega \Sigma X$ since homotopy inverse limits commute. Applying Σ to both sides and using the fact that $\Sigma \Omega$ and $\Omega \Sigma$ are the identity up to homotopy in spectra, we have $\text{holim } \Sigma X \simeq \Sigma \text{holim } \Omega \Sigma X \simeq \Sigma \text{holim } X$. A similar proof shows Ω commutes with hocolim . \square

3.3 Excisive Functors

One of the central notions in Goodwillie calculus is that of “excision”. Generally speaking, an excisive functor takes co-Cartesian cubes to Cartesian cubes. The most common example of an excisive functor is a generalized homology theory. Unfortunately, beginning from the usual axiom for excision, establishing this involves a few details, so we delay it until Section 3.5. Excisive functors in Goodwillie calculus are roughly analogous to polynomial functions; one weakness of this analogy will be noted in Section 3.6.

Technically speaking, an n -excisive functor takes strongly co-Cartesian $(n+1)$ -cubes to Cartesian cubes. Many functors occurring in nature are not excisive; for example, in the category of spaces, neither the identity functor nor Ω^k is n -excisive for any n . However, all good functors satisfy a property known as “stable excision”. The condition of stable excision is a way of codifying to what extent a generalization of the Blakers-Massey theorem holds, so its definition is strongly reminiscent of that theorem. Given an $(\mathbf{n}+1)$ -cube \mathcal{X} in which the map $\mathcal{X}(\emptyset) \rightarrow \mathcal{X}(\{i\})$ is k_i -connected, a functor F is said to be stably n -excisive if the cube $F\mathcal{X}$ is $(\sum k_i - c)$ -Cartesian, for a constant c independent of \mathcal{X} . The stable excision condition with constant c is known as “ $E_n(c)$ ”. On occasion, one

can only guarantee stable excision if the maps are of sufficient connectivity (all $k_i \geq \kappa$); that is also good enough for the machinery of calculus to work, and this condition is known as “ $E_n(c, \kappa)$ ”.

In order to create an n -excisive functor out of a stably n -excisive functor, Goodwillie introduces the functor T_n defined on the category of functors, which takes a functor F satisfying $E_n(c)$ and produces a new functor $T_n F$ satisfying $E_n(c-1)$. This functor is equipped with a natural transformation $F \rightarrow T_n F$, so one can take the colimit to produce an excisive functor (which is what $E_n(-\infty)$ means).

Let us now define $T_n F$. Let $[n]$ denote the set $\{0, 1, \dots, n\}$ with basepoint 0 (this has the same cardinality as the set with the same notation used in the simplicial category), and let $A * B$ denote the topological join of two spaces (considered as *unpointed* spaces). Note that $X * \emptyset = X$, and $X * [0] = CX$, and in general $X * [n]$ is equivalent to $\bigvee^n SX$ — n copies of the unreduced suspension of X . It is easy to see that the $[n]$ -cube $U \mapsto U$ is strongly co-Cartesian. Similarly, for a fixed space X , the $[n]$ -cube

$$\mathcal{X}(U) = X * U \quad (3.10)$$

is strongly co-Cartesian. Since $X * \emptyset = X$, the initial object of this cube is X . This cube without the initial object gives rise to the functor $T_n F$:

$$T_n F(X) = \operatorname{holim}_{U \in \mathcal{P}_0([n])} F(X * U) \quad (3.11)$$

The map from the initial object ($= F(X)$) to the homotopy inverse limit of the punctured cube ($= T_n F(X)$) gives the natural transformation $F \rightarrow T_n F$. A bound on connectivity of this map can be deduced; if X is an r -connected space, then the map $X \mapsto X * [0] = CX \simeq *$ is $(r+1)$ -connected. If F satisfies $E_n(c)$, then the cube is at least $((n+1)(r+1) - c)$ -Cartesian. Furthermore, all of the vertices in the cube used to define $T_n F$ have connectivity at least $r+1$, so iterating the T_n construction produces a map $T_n F \rightarrow T_n T_n F$ that is even more highly connected. The limit $P_n F(X) = \operatorname{colim}_k T_n^k F(X)$ is the universal n -excisive approximation to F , and the map $F(X) \rightarrow P_n F(X)$ is $((n+1)(r+1) - c)$ -connected.

Goodwillie [16] shows that if F is m -excisive or analytic (see Section 3.6), then iterating $T_n F$ produces an n -excisive functor. Actually, as he later established, the functor $P_n F$ is always n -excisive.

Theorem 3.12. ([16]) *If F is m -excisive for some m , then $P_n F$ is the universal n -excisive approximation to F .*

The functor T_n commutes with holim since homotopy inverse limits commute. The functor P_n commutes with finite homotopy inverse limits, since filtered $\operatorname{hocolim}$ and finite holim commute (up to equivalence).

The most important theorem about the structure of the Taylor tower gives information about the fiber $D_n F$ of the map $P_n F \rightarrow P_{n-1} F$: the space $D_n F(X)$ turns out to be an infinite loop space.

Theorem 3.13. ([16]) *If F is an analytic functor from spaces to spaces that commutes up to equivalence with filtered colimits of finite complexes (i.e., satisfies the limit axiom (5.1)), then the functor $D_n F$ is an n -homogeneous functor given by*

$$D_n F(X) = \Omega^\infty(C_n \wedge_{h\Sigma_n} X^{\wedge n}),$$

where C_n is some spectrum with a Σ_n action, and the smashing over $h\Sigma_n$ denotes taking homotopy orbits.

For any functor F , the $D_n F$ are called the “layers” of the Taylor tower, and the associated C_n are called the “coefficient spectra”.

As part of working out a theory of Postnikov invariants for his Taylor tower, Goodwillie shows that there is actually a functorial delooping of the derivatives so the usual fibration sequence $D_n \rightarrow P_n \rightarrow P_{n-1}$ can be delooped once:

Theorem 3.14. ([16]) *If F is a reduced, analytic functor from spaces to spaces, then the map $P_n F \rightarrow P_{n-1} F$ is part of a fibration sequence*

$$P_n F \rightarrow P_{n-1} F \rightarrow \Omega^{-1} D_n F,$$

where $\Omega^{-1} D_n F$ is a homogeneous n -excisive functor whose loop space is necessarily $D_n F$.

3.4 Example: $P_1 F(X)$

When F is a reduced functor, the construction for $P_1 F(X)$ is the stabilization of $F(X)$. First, let us compute $T_1 F(X)$ and the map $F(X) \rightarrow T_1 F(X)$. The diagram to consider to construct $T_1 F(X)$ is:

$$\begin{array}{ccc} F(X) & \longrightarrow & F(X * [0]) \\ \downarrow & & \downarrow \\ F(X * [0]) & \longrightarrow & F(X * [1]) \end{array}$$

The homotopy inverse limit of the punctured cube is easy to understand once we recall $X * [0] = CX \simeq *$ and $X * [1] \simeq \Sigma X$.

$$\begin{aligned} \operatorname{holim}(F(X * [0]) \leftarrow F(X * [1]) \rightarrow F(X * [0])) &\simeq \operatorname{holim}(* \leftarrow F(\Sigma X) \rightarrow *) \\ &\simeq \Omega F(\Sigma X) \end{aligned}$$

The map is then the stabilization map $F(X) \rightarrow \Omega F(\Sigma X)$. Taking the limit as this process is iterated produces $P_1 F(X) \simeq \operatorname{colim} \Omega^n F(\Sigma^n X)$.

3.5 Example: Homology Theories Are 1-excisive

In this section we explain in detail how the excision axiom for homology corresponds to 1-excisiveness. One statement of the excision axiom for homology theories is:

Given the pair (X, A) and an open set $U \subset X$ such that $\overline{U} \subset \text{int}(A)$, the inclusion $(X - U, A - U) \hookrightarrow (X, A)$ induces an isomorphism $H_*(X - U, A - U) \cong H_*(X, A)$. [10, IV.6, p. 183]

The data given here corresponds to the existence of a certain (strongly) co-Cartesian cube:

$$\begin{array}{ccc} A - U & \longrightarrow & A \\ \downarrow & & \downarrow \\ X - U & \longrightarrow & X \end{array}$$

Applying the singular chains functor C_* , we have a cube of chain complexes (whose homology may be thought of as the homotopy groups of the functor $Y \mapsto \mathbf{HZ} \wedge Y$, as noted in Section 2.7). The relative homology groups are the mapping cones of the maps $H_*(A - U) \rightarrow H_*(X - U)$ and $H_*(A) \rightarrow H_*(X)$. The assertion that they are isomorphic is equivalent to asserting that the cube is co-Cartesian (because a cube is co-Cartesian if and only if the iterated homotopy cofiber (=mapping cone) of the cube is contractible). By Lemma 3.7, a cube of spectra is co-Cartesian if and only if it is Cartesian, so the resulting cube is Cartesian as well. Hence H_* takes this co-Cartesian cube to a Cartesian cube of spectra. It remains to show that H_* takes all (strongly) co-Cartesian 2-cubes to Cartesian cubes. By replacing an arbitrary co-Cartesian cube with a weakly equivalent CW-cube (all spaces CW complexes, all maps CW inclusions), we obtain nice inclusion maps. Let \mathcal{X} denote our CW 2-cube. Putting $X = \mathcal{X}(\{1, 2\})$, $A = \mathcal{X}(\{1\})$, and $U = A - \mathcal{X}(\emptyset)$, it is easy to check that $\mathcal{X}(\{2\}) = X - U$. At this point, we must recall that despite the statement of the axiom given above, it is sufficient to require that the pair $(A, A - U)$ be an NDR pair, or that $A - U \rightarrow A$ be a cofibration. Since all CW inclusions are cofibrations, our cube satisfies this hypothesis, reducing the general case to the one we have worked out previously.

3.6 Analytic Functors

If the excisive functors of Goodwillie calculus are analogous to polynomial functions in ordinary calculus, analytic functors are analogous to functions with whose Taylor series converge in some disk about the origin.

An informal statement of analyticity is this: a functor is r -analytic if the coefficient spectra C_n that compose its layers have a connectivity that tends to

$-\infty$ with slope roughly $\geq -rn$ for some c independent of n . In some sense, this means that for spaces of connectivity $\geq r$, the individual “layers” C_n can be distinguished. We will give a rigorous definition of analyticity and later, in Section 6.1 prove that it has the properties of this informal definition.

Definition 3.15 (analytic). Formally, a functor F is r -analytic if there exists a constant c depending only on F such that F satisfies $E_n(rn - c)$ for each n .¹

The immediate consequence of this definition is that, according to the computation following (3.11), if X is r -connected, the map $F(X) \rightarrow P_n F(X)$ is $(n + r + c + 1)$ -connected. The critical trait is that in this case, the connectivity of the map $F(X) \rightarrow P_n F(X)$ increases with n . In this case, the Taylor tower of F is said to converge, since in the inverse limit, the map $F \rightarrow \text{holim } P_n F$ is an equivalence (∞ -connected).

Arguments about analytic functors frequently make use of asymptotic estimates such as the preceding one. To make the essence of these estimates clearer, we will omit the irrelevant constants and use the phrase “approximately n -connected” to mean “there exists a constant c independent of the variables appearing, such that the map is at least $(n + c)$ -connected”. The following lemma is a good example of this.

Lemma 3.16. *If F is r -analytic and X is $(m - 1)$ -connected (for example $X = S^m$), then the map $F(X) \rightarrow P_n F(X)$ is approximately $n(m - r)$ -connected.*

Proof. This is a direct computation. By hypothesis, F satisfies $E_n(rn)$ — note that we have omitted the c — and hence takes strongly co-Cartesian squares of the form used in constructing T_n (3.10) to k -Cartesian squares, where $k = -rn + m(n + 1) = nm - rn + m = n(m - r) + m \approx n(m - r)$. If it were possible for m to be negative, we would want to be more careful about ignoring the $+m$ to get a lower bound. \square

In this work, the statement “ F is analytic” means “ F is r -analytic for some r ”. This is a different usage from that of Goodwillie’s first paper on calculus [17], where “analytic” means “1-analytic”, but consistent with later usage [16].

3.7 Technical Lemmas

In this section, we record some technical lemmas about Cartesian cubes and excisive functors that we will need later.

The first lemma is that given a Cartesian cube of cubes, the cube resulting from taking fibers of the inner cubes is still Cartesian. We begin by recalling Proposition 0.2 from [18], which we will use to perform our decomposition of the homotopy inverse limit.

¹Goodwillie only requires the weaker condition $E_n(rn - c, r + 1)$ be satisfied.

Proposition 3.17 ([18, Proposition 0.2]). *If \mathcal{A} is covered by \mathcal{A}_1 and \mathcal{A}_2 in the sense that the nerve of \mathcal{A} is the union of the nerves of \mathcal{A}_1 and \mathcal{A}_2 , then for any functor F from \mathcal{A} to unbased spaces, the diagram of fibrations*

$$\begin{array}{ccc} \operatorname{holim}(F) & \longrightarrow & \operatorname{holim}(F|_{\mathcal{A}_1}) \\ \downarrow & & \downarrow \\ \operatorname{holim}(F|_{\mathcal{A}_2}) & \longrightarrow & \operatorname{holim}(F|_{(\mathcal{A}_1 \cap \mathcal{A}_2)}) \end{array}$$

is a pullback square. \square

Lemma 3.18. *Let \mathcal{X} be a $(S \amalg T)$ -cube, regarded as an S -cube \mathcal{Y} of T -cubes $\mathcal{Y}(U)$, for $U \subset S$. Let $\tilde{\mathcal{Y}}$ be the S -cube of total fibers of the T -cubes $\mathcal{Y}(U)$. If \mathcal{X} is Cartesian, then $\tilde{\mathcal{Y}}$ is Cartesian.*

Proof. Recall that one may compute $\tilde{\mathcal{Y}}(U)$ as the fiber of the map

$$\mathcal{Y}(U)(\emptyset) \rightarrow \operatorname{holim}_{V \in \mathcal{P}_0(T)} \mathcal{Y}(U)(V).$$

The cube $\tilde{\mathcal{Y}}$ is Cartesian if the map

$$\tilde{\mathcal{Y}}(\emptyset) \rightarrow \operatorname{holim}_{U \in \mathcal{P}_0(S)} \tilde{\mathcal{Y}}(U) \tag{3.19}$$

is an equivalence. Consider the cube:

$$\begin{array}{ccc} \mathcal{Y}(\emptyset)(\emptyset) & \longrightarrow & \operatorname{holim}_{U \in \mathcal{P}_0(S)} \mathcal{Y}(U)(\emptyset) \\ \downarrow & & \downarrow \\ \operatorname{holim}_{V \in \mathcal{P}_0(T)} \mathcal{Y}(\emptyset)(V) & \longrightarrow & \operatorname{holim}_{\substack{U \in \mathcal{P}_0(S) \\ V \in \mathcal{P}_0(T)}} \mathcal{Y}(U)(V) \end{array} \tag{3.20}$$

The fibers of this cube in the vertical direction are the functors of $\tilde{\mathcal{Y}}$ that we are interested in. We will show that if \mathcal{X} is Cartesian, then this cube is Cartesian, and hence that (3.19) is an equivalence.

Let \mathcal{A}_1 be the full subcategory of $\mathcal{P}_0(S \amalg T)$ generated by $\{(U \times V) \mid U \in \mathcal{P}_0(S), V \subset T\}$, and similarly let \mathcal{A}_2 be generated by $\{(U \times V) \mid U \subset S, V \in \mathcal{P}_0(T)\}$. Using Proposition 3.17, the following cube is a homotopy pullback:

$$\begin{array}{ccc} \operatorname{holim}_{\mathcal{P}_0(S \amalg T)} \mathcal{Y} & \longrightarrow & \operatorname{holim}_{\mathcal{A}_1} \mathcal{Y} \\ \downarrow & & \downarrow \\ \operatorname{holim}_{\mathcal{A}_2} \mathcal{Y} & \longrightarrow & \operatorname{holim}_{\mathcal{A}_1 \cap \mathcal{A}_2} \mathcal{Y} \end{array}$$

We can then recognize

$$\begin{aligned}
\operatorname{holim}_{U \in \mathcal{P}_0(S)} \mathcal{Y}(U)(\emptyset) &\simeq \operatorname{holim}_{U \in \mathcal{P}_0(S)} \operatorname{holim}_{V \in \mathcal{P}(T)} \mathcal{Y}(U)(T) \\
&\simeq \operatorname{holim}_{\substack{(U \times V) \in \mathcal{P}_0(S \amalg T) \\ U \in \mathcal{P}_0(S), V \subset T}} \mathcal{Y}(U)(T) \\
&\simeq \operatorname{holim}_{\mathcal{A}_1} \mathcal{Y},
\end{aligned}$$

to identify the upper right hand corner of (3.20). Similarly, the lower left corner is $\operatorname{holim}_{\mathcal{A}_2} \mathcal{Y}$ and the lower right corner is $\operatorname{holim}_{\mathcal{A}_1 \cap \mathcal{A}_2} \mathcal{Y}$. Hence $\operatorname{holim}_{\mathcal{P}_0(S \amalg T)} \mathcal{Y}$ is actually equivalent to the homotopy pullback of the lower right hand corner of the cube in (3.20).

If \mathcal{X} is Cartesian, that is exactly the assertion that the map

$$\mathcal{Y}(\emptyset)(\emptyset) \rightarrow \operatorname{holim}_{\mathcal{P}_0(S \amalg T)} \mathcal{Y}$$

is an equivalence, and $\operatorname{holim}_{\mathcal{P}_0(S \amalg T)} \mathcal{Y}$ is the homotopy pullback of the lower right of (3.20), so (3.20) is actually Cartesian, as desired. \square

The next lemma is that given mild conditions, n -excisiveness is preserved by extensions along fibrations.

Lemma 3.21. *Let $A \rightarrow B \rightarrow C$ be a fibration of homotopy functors from spaces to spaces. If A and C are n -excisive, and furthermore either:*

1. *B takes connected values; or*
2. *$\pi_0 B$ and $\pi_0 C$ lift to functors to groups, and the natural transformation $\pi_0 B \rightarrow \pi_0 C$ is a surjective group homomorphism,*

then B is n -excisive.

Proof. Let \mathcal{Y} be a strongly co-Cartesian $(n+1)$ -cube. By the hypothesis that A and C are n -excisive, the left and right vertical maps are equivalences:

$$\begin{array}{ccccc}
A\mathcal{Y}(\emptyset) & \longrightarrow & B\mathcal{Y}(\emptyset) & \longrightarrow & C\mathcal{Y}(\emptyset) \\
\downarrow \simeq & & \downarrow & & \downarrow \simeq \\
\operatorname{holim}_{\mathcal{P}_0(\mathbf{n}+1)} A\mathcal{Y} & \longrightarrow & \operatorname{holim}_{\mathcal{P}_0(\mathbf{n}+1)} B\mathcal{Y} & \longrightarrow & \operatorname{holim}_{\mathcal{P}_0(\mathbf{n}+1)} C\mathcal{Y}
\end{array}$$

We can then use the Five Lemma on the long exact sequences of the two fibrations to conclude that the middle map is an isomorphism on π_m , for $m \geq 1$. It remains to handle π_0 . By hypothesis, $\pi_0 B\mathcal{Y} \rightarrow \pi_0 C\mathcal{Y}$ is a surjection. A diagram chase then shows that the middle vertical map is an isomorphism on π_0 . Therefore B is n -excisive. \square

Chapter 4

Left Kan Extensions

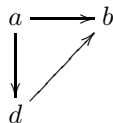
The left Kan extension provides a natural way of extending a functor defined on a category \mathcal{C} to one defined on a larger category \mathcal{D} . In complete generality, extensions of functors do not always exist, but in our setting, where we are dealing with subcategories of topological spaces or spectra, there is no problem.

4.1 Strict Left Kan Extension

To understand how the left Kan extension functions, first let us begin by working in a simpler setting. Suppose V is a sub-vector space of W , and we are given a function f defined on V . The analogous question is, “Does there exist an \tilde{f} defined on all of W that agrees with f on V ?” In the case of vector spaces, the answer is clearly yes; we can extend by zero (or anything else) on the orthogonal complement to V . Evidently, this method is heavily dependent on the existence of an inner product.

The case of categories and functors is not quite so simple. If a morphism $\alpha : A \rightarrow B$ in \mathcal{C} factors through an object D in \mathcal{D} (after inclusion), then any functor that sends all D to zero must also send α to zero. But this may not be what our original functor “ f ” does to α . Because of this complication, we need a more clever way of extending a functor from \mathcal{C} to one on \mathcal{D} .

First, consider the root of the problem: let $a, b \in \mathcal{C}$, and let $d \in \mathcal{D}$, and let F denote the functor on \mathcal{C} that we are trying to extend to \mathcal{D} . Suppose that there is a single morphism from a to b , and that it factors through d .



One candidate for $\tilde{F}(d)$ that will make the diagram commute is $\tilde{F}(d) = F(a)$. We can then collapse the morphism $a \rightarrow d$ to the identity, and let $\tilde{F}(d \rightarrow b) = F(a \rightarrow b)$.

To extend this scheme to the situation where there is more than one morphism from a to d , we can let $\tilde{F}(d) = F(a) \times \text{Hom}(a, d)$. Then for each $j : a \rightarrow d$, we can let $\tilde{F}(j)$ send $F(a)$ to $(F(a), j) \hookrightarrow F(a) \times \text{Hom}(a, d)$. To do this consistently for all objects of \mathcal{D} , take the coproduct over all objects of \mathcal{C} . Then there is a small matter that for objects in \mathcal{C} , the new function \tilde{F} will be too large, since it consists of at least $F(C) \times \text{Hom}(C, C)$, so one must divide out the morphisms of \mathcal{C} by identifying $(F(f)F(C), 1)$ with $(F(C), f)$. In general let \tilde{F} be the functor given by taking the coproduct over all objects in \mathcal{C} of $F(C) \times \text{Hom}(C, -)$, and then equalizing out the morphisms in \mathcal{C} ; that is, consider

$$\bigvee_{C \in \mathcal{C}} F(C) \times \text{Hom}(C, -),$$

and for every morphism $f : C \rightarrow C'$ in \mathcal{C} , identify $(F(C), f)$ with $(F(C'), 1)$. This has the effect of forcing $\tilde{F}(C')$ to be equal to $F(C')$ since every element of $f \in \text{Hom}(C, C')$ gives rise to an identification $(F(C), f)$ with $(F(C'), 1)$, so every element of the coproduct is either empty (if there are no morphisms $C \rightarrow C'$) or identified with $F(C')$. Another way of writing this is

$$\tilde{F} = F(\mathcal{C}) \otimes_{\mathcal{C}} \text{Hom}(\mathcal{C}, -),$$

where the coproduct over all $C \in \mathcal{C}$ is implicit in the meaning of $\otimes_{\mathcal{C}}$. This is the left Kan extension, also known as a “coend”. It is sometimes written $\int^c F(c) \times \text{Hom}(c, -)$, which is useful for understanding interchanges of limiting processes, but (in my opinion) makes less familiar the properties we are most interested in.

We will use the notation $L_I F$ to denote the left Kan extension of F along I . When I is the inclusion of a subcategory $\mathcal{C} \rightarrow \mathcal{D}$, we will use $L_{\mathcal{C}} F$ instead.

Theorem 4.1. ([20, §X.3, Corollary 3, p. 235]) *If $I : \mathcal{C} \rightarrow \mathcal{D}$ is full and faithful, then the natural transformation $L_I(F) \circ I \rightarrow F$ is an isomorphism.*

For more information on Kan extensions and coends, see [20, Chapter X].

4.2 Homotopy Invariant Left Kan Extension

We consider a simplicial resolution of the strict left Kan extension for two reasons: we want to guarantee that we have a homotopy functor, and the layers of the simplicial resolution are easier to understand than strict left Kan extension itself. For the remainder of this paper, “left Kan extension” will mean the simplicial resolution of the left Kan construction, as defined in this section.

Definition 4.2 (Left Kan extension). Let F be a functor from spaces to spaces. The homotopy-invariant left Kan extension $L_{\mathcal{C}} F$ of F over a subcategory \mathcal{C} of the category of spaces \mathcal{D} is given by the realization of the simplicial functor to

spaces:

$$[n] \mapsto \bigvee_{(C_0, \dots, C_n)} F(C_0) \wedge (\text{Hom}_{\mathcal{C}}(C_0, C_1) \times \cdots \times \text{Hom}_{\mathcal{D}}(C_n, -))_+. \quad (4.3)$$

The coproduct is taken over all $(C_0, \dots, C_n) \in \mathcal{C}^{\times n}$. (Recall that the product of any space with an empty space is the empty space, so this is really the (continuous) nerve of the category in disguise.) We use the construction $F \wedge (-)_+$ rather than $F \times (-)$ so that the construction is immediately applicable in the case of functors from spaces to spectra as well. Recall that spaces is a topological category, and we use the mapping space Hom .

Lemma 4.4. *The left Kan extension given by (4.3) is a homotopy functor if \mathcal{C} is a full subcategory of spaces whose objects are cofibrant.*

Proof. Realizations take levelwise weakly equivalent object to weakly equivalent objects, so we need only show that in each dimension our simplicial functor is a homotopy functor. This consists of tracing through to verify that the conditions mentioned in Section 2.7 hold. A discussion of each of the pieces of this argument occurs on page 14.

Each dimension consists of a coproduct functors; this is homotopy invariant if after evaluation all of the spaces involved have nondegenerate basepoints. We have made a blanket assumption to this effect since all functors to spaces can be (functorially) made to take values in spaces with nondegenerate basepoints by adding a “whisker” if necessary. Similarly, the smash product is a homotopy functor if all spaces involved have nondegenerate basepoints. On the right side of the smash product, we have a product of constant functors

$$\text{Hom}(C_0, C_1) \times \cdots \times \text{Hom}(C_{n-1}, C_n)$$

with the functor $\text{Hom}(C_n, X)$. Constant functors are homotopy invariant, of course, and $\text{Hom}(C_n, X)$ is homotopy invariant because C_n is a cofibrant space (CW complex) by hypothesis. The product of homotopy functors is a homotopy functor, so we are done. \square

As with the strict left Kan extension, $L_n F$ is equipped with a map (natural transformation) to F given by mapping

$$F(C_0) \wedge \left(C_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} C_n \xrightarrow{\beta} X \right)_+$$

to

$$F(\beta \alpha_{n-1} \cdots \alpha_1) : F(C_0) \rightarrow F(X).$$

Evidently, given a map $X \xrightarrow{f} Y$, we have a map $L_{\mathcal{C}} F(X) \rightarrow L_{\mathcal{C}} F(Y)$ given by sending the simplex $C_0 \xrightarrow{\alpha_1} C_1 \rightarrow \cdots \rightarrow C_n \xrightarrow{\beta} X$ to $C_0 \xrightarrow{\alpha_1} C_1 \rightarrow \cdots \rightarrow C_n \xrightarrow{f \circ \beta} Y$, and this is compatible with the map to F , since $F(f)F(\beta \alpha_n \cdots \alpha_1) = F(f \beta \alpha_n \cdots \alpha_1)$.

For this map to be continuous with respect to the topology on the Hom sets requires F to be a continuous functor. To emphasize the topology, we will write Map for Hom here. (But recall that in this work they are both the same — both are topologized.) Recall that a functor F is continuous if given spaces A and B , the map of spaces $\text{Map}(A, B) \rightarrow \text{Map}(FA, FB)$ sending f to Ff is continuous. The map from the left Kan extension arises from the composition of this map with the evaluation map:

$$\begin{array}{c} F(C) \wedge \text{Map}(C, X)_+ \\ \downarrow 1 \wedge F \\ F(C) \wedge \text{Map}(FC, FX)_+ \\ \downarrow \text{eval} \\ F(X) \end{array}$$

The main interesting property of the left Kan extension is that it agrees with the original functor on the category \mathcal{C} up to equivalence. This result is the analog of Theorem 4.1 for the homotopy invariant left Kan extension.

Proposition 4.5. *Let F be a functor from spaces to spaces. Consider the left Kan extension $L_{\mathcal{C}}F$, where \mathcal{C} is a full subcategory of spaces, and let $C \in \text{Obj}(\mathcal{C})$. Then the natural transformation from $L_{\mathcal{C}}F(C)$ to $F(C)$ is an equivalence.*

Proof. This is a general fact about nerves of categories with terminal objects. Consider the general portion of the coproduct in dimension n given by

$$F(C_0) \wedge (C_0 \xrightarrow{\alpha_1} C_1 \rightarrow \cdots \rightarrow C_n \rightarrow C).$$

Iterating the commutative diagram

$$\begin{array}{ccccc} F(C_0) & \wedge & C_0 & \xrightarrow{\alpha_1} & C_1 & \longrightarrow & \cdots & \longrightarrow & C_n & \xrightarrow{\beta} & C \\ \downarrow F(\alpha_1) & & \downarrow \alpha_1 & & \downarrow & & & & \downarrow \beta & & \downarrow = \\ F(C_1) & \wedge & C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots & \xrightarrow{\beta} & C & \xrightarrow{=} & C \end{array}$$

gives a homotopy from $F(C_0) \wedge (\text{Hom}(C_0, C_1) \times \cdots \times \text{Hom}(C_n, C))_+$ to $F(C) \wedge (\text{id}_C \times \cdots \times \text{id}_C)_+$. The latter is $F(C) \wedge S^0 \cong F(C)$. (The hypothesis of being a full subcategory is needed to write the β on the second line, since there it is required to be an element of $\text{Hom}_{\mathcal{C}}(C_n, C)$ instead of just $\text{Hom}_{\mathcal{D}}(C_n, C)$.) \square

The particular left Kan extensions we are interested in commute with realizations of simplicial k -connected spaces, for large enough k , because there is a bound on the dimension of the objects in the subcategory being extended along.

Lemma 4.6. *The functor $\text{Map}(S^n, -)$ commutes with realizations of simplicial $(n-1)$ -connected spaces.*

Proof. By adjunction, $\text{Map}(S^n, X) \cong \text{Map}(S^{n-1}, \text{Map}(S^1, X)) \cong \text{Map}(S^{n-1}, \Omega X)$, and ΩX has connectivity one less than X , so by induction we need only show that $\text{Map}(S^1, -)$ commutes with realizations of simplicial connected spaces. Waldhausen's lemma (Lemma 2.10) implies this, since it shows that if all X_i are connected, then both $\Omega\|X.\|$ and $\|\Omega X.\|$ are equivalent to the homotopy fiber of the map $0 \rightarrow \|X.\|$. \square

Corollary 4.7. *The functor $\text{Map}(\bigvee S^n, -)$ commutes with realizations of simplicial $(n-1)$ -connected spaces.*

Proof. We know $\text{Map}(\bigvee S^n, -) \cong \prod \text{Map}(S^n, -)$, and products commute with realizations. \square

Corollary 4.8. *Let K be a finite CW complex of dimension n . The functor $\text{Map}(K, -)$ commutes with realizations of simplicial $(n-1)$ -connected spaces.*

Proof. The result is true for $K = *$. We proceed by induction, showing that you can add one cell and the result still holds. Suppose $\text{Map}(K', -)$ commutes with realizations of simplicial $(n-1)$ -connected spaces. Suppose that a $(k+1)$ -cell is added to K' along attaching map α to produce K .

$$\begin{array}{ccc} S_+^k & \xrightarrow{\alpha} & K' \\ \downarrow & & \downarrow \\ D_+^{k+1} & \longrightarrow & K \end{array}$$

Applying the functor $\text{Map}(-, X_i)$ to this co-Cartesian square produces a Cartesian square:

$$\begin{array}{ccc} \text{Map}(K, X_i) & \longrightarrow & \text{Map}(D_+^{k+1}, X_i) \\ \downarrow & & \downarrow \\ \text{Map}(K', X_i) & \xrightarrow{\alpha^*} & \text{Map}(S_+^k, X_i) \end{array}$$

Hence we have a 2-cube of simplicial spaces that is levelwise Cartesian. We would like to conclude that after realization, this is still a Cartesian square, so both $\text{Map}(K, \|X.\|)$ and $\|\text{Map}(K, X.)\|$ are equivalent to the inverse limit over the rest of the cube:

$$\|\text{Map}(K, X.)\| \simeq \text{holim} (\|\text{Map}(D_+^{k+1}, X.)\| \rightarrow \|\text{Map}(S_+^k, X.)\| \leftarrow \|\text{Map}(K', X.)\|)$$

which, by the induction hypotheses, is

$$\begin{aligned} &\simeq \text{holim} (\text{Map}(D_+^{k+1}, \|X.\|) \rightarrow \text{Map}(S_+^k, \|X.\|) \leftarrow \text{Map}(K', \|X.\|)) \\ &\simeq \text{Map}(K, \|X.\|). \end{aligned}$$

When k is at most the connectivity of X_i (that is, $k \leq n-1$, so the cell of dimension $k+1$ being attached has dimension $k+1 \leq n$, which is our

hypothesis on $\dim K$), both of the spaces $\text{Map}(S_+^k, X_i)$ and $\text{Map}(D_+^k, X_i)$ are connected. In this circumstance, they trivially satisfy the π_* -Kan condition, so by Theorem 2.12, the square is in fact Cartesian after realization. \square

Proposition 4.9. *Let F be a functor from spaces to spaces. Let \mathcal{C} be a subcategory of CW spaces whose objects have dimension at most a . Then $L_{\mathcal{C}}F$ commutes with realizations of $(a-1)$ -connected simplicial spaces.*

Proof. In (4.3), we see that everything but $\text{Hom}(C_n, -)$ commutes with realizations of X with no conditions. The fact that $\text{Hom}(C_n, -)$ commutes with realizations of simplicial $(a-1)$ -connected spaces is the content of Corollary 4.8. \square

Lemma 4.10. *Let \mathcal{C} be the full subcategory of pointed spaces whose objects are finite coproducts of $S^0: \bigvee^k S^0$ for $k = 0, 1, \dots$. If X is a simplicial set, and F is a functor from spaces to nondegenerately based spaces (as all of our spaces are assumed to be), then the simplicial space $[k] \mapsto L_{\mathcal{C}}F(X_k)$ is good (2.2).*

Proof. From Lemma 2.5, item 7, we know that when X is good, then so is the mapping space $\text{Hom}(C_n, X)$. Following the construction of the left Kan extension (Definition 4.2), Lemma 2.5, item 3, shows that the product with the constant space

$$\text{Hom}(C_0, C_1) \times \cdots \times \text{Hom}(C_{n-1}, C_n)$$

is still a good space. Adding a disjoint basepoint is still good (item 1 in the same lemma), as smashing with a space with a nondegenerate basepoint (item 4), and taking the coproduct over all n -tuples (C_0, \dots, C_n) (by item 6). Finally, the realization in the direction internal to the left Kan extension still produces a good simplicial space by item 7. \square

4.3 Defining Additive Calculus From The Left Kan Extension

Let L denote the left Kan extension (4.2) over the full subcategory of pointed spaces generated by finite coproducts of S^0 (including the empty coproduct, $*$). Let F_X denote the functor sending Y to $F(X \wedge Y)$. This fixes information about X into the functor, so that the left Kan extension LF_X contains information about the value of F on coproducts of X , not just the value of F on points. The functor LF_X naturally comes equipped with a map to F_X , but because we are taking the left Kan extension over a subcategory that contains S^0 , the unit of the smash product, there is also a map $F(X) \rightarrow (LF_X)(S^0)$ — note the change from F_X to $F(X)$ — given by sending $F(X)$ to the 0-simplex $F(X \wedge S^0) \times 1_{S^0}$. Applying P_n to a left Kan extension LF_X creates a theory that we refer to as n -additivization.

$$P_n^d F(X) = P_n(LF_X)(S^0).$$

The decoration “d” stands for “discrete”, since the functor is defined by a left Kan extension over a discrete subcategory of spaces.

Using the fact that left Kan extensions commute with realizations of appropriately highly connected spaces, we can write one of these functors in another, perhaps more familiar, way. Let us compute $P_1^d F(X)$ for a reduced functor F .

$$P_1^d F(X) = P_1(LF_X)(S^0).$$

As in Section 3.4, this is equivalent to

$$\operatorname{colim}_n \Omega^n LF_X(S^n \wedge S^0).$$

Then S^0 is the identity of the smash product, so this equals

$$\operatorname{colim}_n \Omega^n LF_X(S^n).$$

Proposition 4.9 applied with $a = 0$ implies that LF_X commutes with realizations of all simplicial sets, so this is equivalent to

$$\operatorname{colim}_n \Omega^n \|LF_X(S^n)\|.$$

Since LF agrees with F (up to equivalence) on the category of finite discrete spaces, and each S_i^n is a finite discrete space, this equivalent to

$$\operatorname{colim}_n \Omega^n \|F_X(S^n)\|.$$

Which, by the definition of F_X , shows that

$$P_1^d F(X) \simeq \operatorname{colim}_n \Omega^n \|F(X \wedge S^n)\|.$$

Example 4.11. To work out a particular example, let $F(X) = K(H_2(X), 2)$ be the Eilenberg-MacLane space with $\pi_2 = H_2(X)$. (This is another example of using a topological substitute for the category of abelian groups.) We assert that $LF_X(S^1)$ is the bar construction on $F(X)$, and hence $\Omega LF_X(S^1) \simeq \Omega BF(X) \simeq \Omega K(H_2(X), 3) \simeq F(X)$, so $P_1^d F(X) = F(X)$.

Recall that in our standard simplicial set model for S^1 , there are $n + 1$ simplices in dimension n . That is, the model is $[n] \mapsto \bigvee^n S^0$. Applying $H_2(X \wedge -)$ levelwise, we get $[n] \mapsto H_2(\bigvee^n X)$, which is $\oplus^n H_2(X)$. The face maps are induced by the fold map $S^0 \vee S^0 \rightarrow S^0$. This becomes addition on H_2 , since addition is universal as a map from $A \oplus A \rightarrow A$, for any abelian group A , that restricts to the identity on each component of the coproduct $A \oplus A$. This allows us to identify $H_2(X \wedge S^1)$ with the bar construction $BH_2(X)$ on the abelian group $H_2(X)$. Since the functor $K(-, 2)$ preserves products of abelian groups, $K(H_2(X \wedge S^1), 2)$ is the bar construction on $K(H_2(X), 2)$, so $LF_X(S^1)$ is $BF(X)$, as claimed.

Chapter 5

Excisive Functors from Spaces to Spectra

Functors from spaces to spectra that are n -excisive and satisfy the limit axiom (5.1) are determined by their left Kan extensions over the full subcategory of spaces containing as objects the discrete spaces $[k]$, where k ranges from 0 to n (the degree of the functor). To recollect: we use the notation $[k]$ to denote the space $\bigvee^k S^0$, which has $k + 1$ points. In this section, we write $L_n F$ for the left Kan extension of F along the inclusion of the full subcategory of spaces whose objects are $\{[0], \dots, [n]\}$.

Definition 5.1. A homotopy functor F is said to satisfy the *limit axiom* if F commutes with filtered homotopy colimits of finite complexes. That is, if $\operatorname{hocolim} F(X_\alpha) \simeq F(\operatorname{hocolim} X_\alpha)$ for all filtered systems $\{X_\alpha\}$ of finite complexes, then F satisfies the limit axiom.

The limit axiom is needed to relate the values of F on infinite complexes to the values of F on finite complexes. For instance, there are nontrivial functors like $\operatorname{Map}(-, QS^0)$ that are contractible on all finite complexes; the methods in this section evidently will not be able to say anything about these functors.

For the results in this section, considering functors to spectra is critical. The main way in which we use spectra as the target category is embodied in the following lemma:

Lemma 5.2 (Basic Lemma for Spectra). *If \mathcal{X} is a strongly co-Cartesian S -cube of spectra, with $|S| = n + 1$, and F is an n -excisive functor taking values in spectra, then*

$$F\mathcal{X}(S) \simeq \operatorname{hocolim}_{U \in P_1(S)} F\mathcal{X}(U).$$

That is, $F\mathcal{X}$ is co-Cartesian.

Proof. Recall that $P_1(S)$ is the power set of S with the terminal object removed. Since F is n -excisive, it takes co-Cartesian $(n + 1)$ -cubes to Cartesian cubes. In the category of spectra, Cartesian cubes are also co-Cartesian, so the result follows. \square

5.1 $L_n F$ Is n -excisive

Recall that the functor $L_n F(X)$ is given by the realization of a simplicial spectrum:

$$L_n F(X) = \left| [k] \mapsto \bigvee_{(C_0, \dots, C_k)} F(C_0) \wedge (\text{Hom}(C_0, C_1) \times \dots \times \text{Hom}(C_k, X))_+ \right|. \quad (5.3)$$

We begin by showing that to know $L_n F$ is n -excisive, it is enough to know that each simplicial dimension of $L_n F$ is n -excisive.

Lemma 5.4. *Let F be a simplicial functor from spaces to spectra. If each F_i is n -excisive, then $\|F\|$ is n -excisive.*

Proof. Let \mathcal{X} be a strongly co-Cartesian S -cube of spaces, with $|S| = n + 1$. If $\|F\mathcal{X}\|$ is Cartesian, then $\|F\|$ is n -excisive. Cartesian and co-Cartesian are equivalent notions for spectra, so it suffices to show that $\|F(\mathcal{X})\|$ is co-Cartesian.

Each F_i is n -excisive, so $F_i \mathcal{X}(S) \simeq \text{hocolim}_{U \in P_1(S)} F_i \mathcal{X}(U)$. Applying the realization functor to both sides, and noting that realization is a homotopy colimit and colimits commute, we have $\|F\mathcal{X}(S)\| \simeq \text{hocolim}_{U \in P_1(S)} \|F\mathcal{X}(U)\|$. This shows that $\|F\mathcal{X}\|$ is co-Cartesian, as desired. \square

Proposition 5.5. *If F is a functor from spaces to spectra, then $L_n F$ is n -excisive.*

Proof. Lemma 5.4 shows that it suffices to demonstrate that each level of the simplicial spectrum in (5.3) is n -excisive. Since in the category of spectra, finite coproducts are equivalent to products, and the product of n -excisive functors is n -excisive, we only need to show that the functor

$$F_k(X) = F(C_0) \wedge (\text{Hom}(C_0, C_1) \times \dots \times \text{Hom}(C_k, X))_+$$

is n -excisive. Now for spaces it is easy to see that $(A \times B)_+ \cong A_+ \wedge B_+$, so this can be rewritten as

$$F_k(X) \cong F(C_0) \wedge (\text{Hom}(C_0, C_1)_+ \wedge \dots \wedge \text{Hom}(C_k, X)_+).$$

Using the associativity of smash product (of a space with a spectrum), we have:

$$F_k(X) \cong (F(C_0) \wedge \text{Hom}(C_0, C_1)_+ \wedge \dots \wedge \text{Hom}(C_{k-1}, C_k)_+) \wedge \text{Hom}(C_k, X)_+.$$

This is the smash product of a constant functor (which we will denote \mathbf{C}) to spectra with $\text{Hom}(C_k, X)_+$. The category over which we have taken the left Kan extension consists of finite sets of cardinality at most n , and C_k is one of these sets. The space of maps of a finite set into X is just a product of copies of X ; the space of pointed maps of $[n]$ into X is isomorphic to $X^{\times n}$. In [18,

Example 3.5], Goodwillie shows that $\mathbf{C} \wedge (X_+^{\times n})$ is n -excisive for any spectrum \mathbf{C} . Therefore, $L_n F$ is n -excisive. \square

Lemma 5.6. *For any functor F from spaces to spaces or spectra, and any subcategory \mathcal{C} of spaces whose objects are finite CW-complexes, the left Kan extension $L_{\mathcal{C}} F$ satisfies the limit axiom (5.1).*

Proof. We need to show that if Y is equivalent to the filtered homotopy colimit of its finite subcomplexes $\{Y_\alpha\}$, then $L_{\mathcal{C}} F(Y) \simeq \text{hocolim } L_{\mathcal{C}} F(Y_\alpha)$. If C_n is a finite complex, then its image is compact, and hence lies inside some finite Y_α , so $\text{Hom}(C_n, -)$ commutes with filtered homotopy colimits. In the definition of the homotopy left Kan extension (Equation (4.3)), the only term that involves Y or Y_α is $\text{Hom}(C_n, -)$, where $C_n \in \text{Obj}(\mathcal{C})$, so $L_{\mathcal{C}} F$ satisfies the limit axiom because $\text{Hom}(C_n, -)$ does for all $C_n \in \text{Obj}(\mathcal{C})$. \square

Corollary 5.7. *If F is a functor from spaces to spectra, then the functor $L_n F$ satisfies the limit axiom.*

Proof. The sets $[n]$ are all finite CW complexes, so this is immediate from Lemma 5.6. \square

5.2 Excisive Functors Are Left Kan Extensions

In this section, we establish that any n -excisive functor from spaces to spectra that satisfies the limit axiom (5.1) commutes with the realization of a simplicial spaces. That is, such a functor F is equivalent to its own left Kan extension $L_n F$.

We begin by establishing the lemma that $L_n F$ and F agree on finite sets.

Lemma 5.8. *If F is an n -excisive functor from spaces to spectra, then for all finite sets X , the map $L_n F(X) \rightarrow F(X)$ is an equivalence.*

Proof. Let $m = |X|$ be the cardinality of X . If $m \leq |[n]|$, then by Proposition 4.5, the map $L_n F(X) \rightarrow F(X)$ is an equivalence. If $m > |[n]|$, then we may assume by induction that the result is true for all smaller m .

Let $S = \{1, 2, \dots, n+1\}$. Define pointed sets W_u for $u \in S$ as follows:

$$W_u = \begin{cases} \{*, u\} & \text{if } u \neq n+1 \\ \{*, n+1, \dots, m-1\} & \text{if } u = n+1 \end{cases}$$

Let \mathcal{X} be the strongly co-Cartesian S -cube given by $\mathcal{X}(U) = \bigvee_{u \in U} W_u$. Note that $\mathcal{X}(S) = \{*, 1, 2, \dots, m-1\}$ has m points, and hence is isomorphic to X .

For $U \subsetneq S$, we have $|\mathcal{X}(U)| < m$, so by our induction hypothesis, $L_n F \mathcal{X}(U) \simeq F \mathcal{X}(U)$ for all $U \in \mathcal{P}_1(S)$. Now $L_n F$ is n -excisive (Proposition 5.5), and F is n -excisive, so the Basic Lemma for Spectra (5.2) shows that both $L_n F \mathcal{X}$ and $F \mathcal{X}$ are co-Cartesian, so we have an equivalence on the terminal vertices as well. That is, $L_n F(X) \simeq F(X)$. \square

Theorem 5.9. *Let F be an n -excisive functor from spaces to spectra that satisfies the limit axiom (5.1), and let $L_n F$ be as defined in (5.3). For any space X , the map $L_n F(X) \rightarrow F(X)$ is a weak equivalence.*

Proof. The functors $L_n F$ and F are both homotopy functors that satisfy the limit axiom, so it is sufficient to establish that the theorem is true when X is the realization of a finite simplicial set. We proceed by induction on the dimension of X , where by dimension we mean: as usual, $\dim(X)$ is the largest k such that X_k contains nondegenerate elements, and

$$\dim(X) = \min\{\dim(X.) \mid X. \text{ finite and } ||X.|| \simeq X\}.$$

Base Case. When $\dim(X) = 0$, the space X is a finite set of points. Lemma 5.8 shows that the map $L_n F(X) \rightarrow F(X)$ is an equivalence for all finite X .

Induction Case: Adding an $(m+1)$ -cell. We now proceed by induction, assuming that if $\dim(X) \leq m$, then $L_n F(X) \simeq F(X)$. To add $(m+1)$ -cells to X , we consider a second induction on the minimal number of $(m+1)$ -cells needed to build X .

To form a space Y by attaching an $(m+1)$ -cell to X along f , one forms the pushout:

$$\begin{array}{ccc} S^m & \longrightarrow & D^{m+1} \\ \downarrow f & & \downarrow \\ X & \longrightarrow & Y \end{array} \quad (5.10)$$

In order to use the n -excisive properties of $L_n F$ and F , we need to blow up this cubical diagram to be of dimension at least $n+1$. We will do this by subdividing D^{m+1} until it has at least $n+1$ simplices of dimension $(m+1)$, and then gluing them into its m -skeleton one by one.

Let D^{m+1} denote the standard $(m+1)$ simplex. Let $R(r)$ be the set of non-degenerate simplices of dimension $(m+1)$ in the r -fold subdivision of D^{m+1} , which we denote $sd_r D^{m+1}$. Choose $r \gg 0$ large enough that $|R(r)| \geq n+1$, and for convenience let $R = R(r)$ for this r . Now form an R -cube \mathcal{D} by gluing these $(m+1)$ -simplices onto the m -skeleton of $sd_r(D^{m+1})$. Explicitly,

$$\mathcal{D}(U) = \left(\bigcup_{r \in U} r \right) \cup \text{Skel}_m sd_r(D^{m+1}).$$

Notice that $\mathcal{D}(R) = sd_r(D^{m+1})$, so this cube expresses the $(m+1)$ -simplex as a pushout of dimension at least $n+1$.

Instead of forming the exact analog of the pushout diagram in (5.10), we replace the space X by another space X' , which is X with the m -skeleton of

$sd_r D^{m+1}$ glued on along the attaching map f .

$$\begin{array}{ccccccc}
S^m & \longrightarrow & \text{Skel}_m sd_r D^{m+1} & \longrightarrow & \cdots & \longrightarrow & sd_r D^{m+1} \\
\downarrow f & & \downarrow f' & & & & \downarrow \\
X & \longrightarrow & X' & \longrightarrow & \cdots & \longrightarrow & Y
\end{array}$$

While X is not equivalent to X' , the space X' still satisfies our induction hypothesis since we have not added any $(m+1)$ -cells. We will not use X itself further in this proof, except to identify the space Y below as X with an $(m+1)$ -cell attached along f .

Define the $S = R\amalg\{*\}$ cube \mathcal{Y} to be the strongly co-Cartesian cube generated by \mathcal{D} and the map $\mathcal{D}(\emptyset) \xrightarrow{f'} X'$.

$$\mathcal{Y}(U) = \begin{cases} \mathcal{D}(U) & \text{if } * \notin U \\ \text{colim} \left(X' \xleftarrow{f'} \mathcal{D}(\emptyset) \rightarrow \mathcal{D}(U - \{*\}) \right) & \text{if } * \in U \end{cases}$$

From its construction, it is evident that $\mathcal{Y}(S) = Y$, where Y is the pushout $Y = \text{colim}(X \xleftarrow{f} S^m \rightarrow D^{m+1})$.

$L_n F$ and F are n -excisive (5.5), and $|S| \geq n+1$, so $L_n F\mathcal{Y}$ and $F\mathcal{Y}$ are co-Cartesian cubes (5.2). Therefore, to show $L_n F\mathcal{Y}(S) \simeq F\mathcal{Y}(S)$ (that is, $L_n F(Y) \simeq F(Y)$), we need only show $L_n F\mathcal{Y}(U) \simeq F\mathcal{Y}(U)$ for $U \subsetneq S$.

All of the non-terminal vertices $\mathcal{D}(U)$ of \mathcal{D} are subdivisions of D^{m+1} with some $(m+1)$ -cells missing. All of these retract relative to their boundary to complexes of dimension m , so they all satisfy our induction hypothesis. Furthermore, this retraction relative to the boundary also shows that $\mathcal{Y}(U \amalg \{*\})$ satisfies the induction hypothesis. Finally, $\mathcal{D}(R) \simeq *$, so on all nonterminal vertices of \mathcal{Y} , we have $L_n F\mathcal{Y}(U) \simeq F\mathcal{Y}(U)$. That is what we needed to establish. \square

Corollary 5.11. *Let F be an n -excisive functor from spaces to spectra satisfying the limit axiom (5.1). Then F commutes with realizations of all simplicial spaces.*

Proof. Theorem 5.9 shows that F is equivalent to a left Kan extension over a subcategory of spaces containing only objects of dimension 0. Then Proposition 4.9 shows that this left Kan extension commutes with realization of (-1) -connected simplicial spaces. But all (nonempty) spaces are (-1) -connected. \square

Chapter 6

Analyticity And Realization

In this chapter we establish the result that analytic functors from spaces to spaces commute with realizations of highly connected spaces, and hence are equivalent to certain left Kan extensions. In order to do this, we also establish properties of analytic functors that show our intuition about the behavior of the coefficient spectra is justified.

6.1 Analytic Functors Have Connective Coefficient Spectra

In this section, we establish the following theorem, which states that an analytic functor has coefficient spectra that are bounded below.

Theorem 6.1. *Let F be a functor with coefficient spectra \mathbf{C}_i (defined only for $i \geq 1$). If F is r -analytic with universal analyticity constant c , so F satisfies $E_n(rn - c)$ for all n , then $\pi_j(\mathbf{C}_{n+1}) = 0$ for $j < c - rn$. In particular, all \mathbf{C}_i are bounded below.*

If “analyticity” is to be a well-behaved concept, we need to prove that if F is analytic, then so is $P_n F$. We do this by showing that if F satisfies $E_m(c)$, then so does $T_n F$. Goodwillie [18] proves that when $m = n$, the functor $T_m F$ actually satisfies at least $E_m(c - 1)$; we will reiterate his argument as part of establishing the fact we are most interested in. We begin by recalling a technical proposition.

Proposition 6.2. *([18, Proposition 1.22]) Let \mathcal{X} be a functor from $\mathcal{P}_0(S)$ to T -cubes of spaces, and write $\mathcal{X}(U, V) = (\mathcal{X}(U))(V)$. For each $U \in \mathcal{P}_0(S)$, let k_U be a constant so that the T -cube $\mathcal{X}(U)$ is k_U -Cartesian. Then the T -cube $V \mapsto \text{holim}(U \mapsto \mathcal{X}(U, V))$ is k -Cartesian with $k = \min\{1 - |U| + k_U\}$. \square*

Proposition 6.2 is immediately applicable to the T_n construction. Our main interest in this is for $n = m + 1$, where the $E_m(c)$ condition satisfied is not improved by T_{m+1} .

Corollary 6.3. *If F satisfies $E_m(c)$, then for $n \leq m + 1$, so does $T_n F$.*

Proof. Let \mathcal{Y} be a strongly co-Cartesian T -cube, with $T = [m]$ and the map $\mathcal{Y}(\emptyset) \rightarrow \mathcal{Y}(\{j\})$ being k_j -connected. Recall that $T_n F(X) = \operatorname{holim}_{U \in \mathcal{P}_0(S)} F(X * U)$, for $S = [n]$, so Proposition 6.2 applies to the consideration of the functor $\mathcal{X}(U, V) \mapsto F(\mathcal{X}(V) * U)$ (which is only defined for $\emptyset \neq U \subset S$). Since F satisfies $E_m(c)$ and $*U$ raises connectivity by one (for $U \neq \emptyset$), the cube $\mathcal{X}(U)$ is k_U -Cartesian, with $k_U = \sum_{j=0}^m (k_j + 1) - c = \sum k_j - c + (m + 1)$. Applying Proposition 6.2, we see that the T -cube $V \mapsto T_n F(\mathcal{Y}(V))$ is k -Cartesian, with $k = \min\{1 - |U| + k_U\}$. As $|U| \leq n + 1$, we know $k \geq 1 - (n + 1) + \sum k_j - c + (m + 1) = 1 - (m - n) + \sum k_j - c$. This shows that $T_n F$ satisfies $E_m(c + m - n - 1)$. Therefore, for $m \geq n - 1$, if F satisfies $E_m(c)$, then so does $T_n F$. \square

This argument can now be used to show that $P_n F$ satisfies the same stable excision condition as F for n -cubes. Of course, applying $P_n F$ to larger cubes results in Cartesian cubes.

Example 6.4. In general, $P_{n+1} F$ may have a better constant $E_n(c)$ than F does. Consider the functor from spaces to spectra given by

$$F(X) = (\mathbf{H}\mathbf{Z} \wedge X) \times (\mathbf{S}^{-2} \wedge \mathbf{H}\mathbf{Z} \wedge X \wedge X).$$

This functor satisfies $E_0(2)$ since it takes (-1) -connected maps to (-3) -connected maps, but $P_1 F(X) = \mathbf{H}\mathbf{Z} \wedge X$ satisfies $E_0(0)$. The functor F is (-1) -analytic with constant $c = -2$; this example shows that the increasing of the constant $E_n(c)$ when passing from F to $P_{n+1} F$ relates to the constant, not the analyticity or radius of convergence.

Corollary 6.5. *If F satisfies $E_n(c)$, then so does $P_{n+1} F$.*

Proof. Recall that $P_{n+1} F = \operatorname{colim}_k (T_{n+1})^k F$. Let \mathcal{X} be a strongly co-Cartesian S -cube, with $S = [n]$, and let k_i denote the connectivity of the map $\mathcal{X}(\emptyset) \rightarrow \mathcal{X}(\{i\})$. Suppose F satisfies $E_n(c)$. By Corollary 6.3, this implies $T_{n+1} F$ satisfies $E_n(c)$, and hence by induction all $(T_{n+1})^k F$ satisfy $E_n(c)$. We need to establish that the colimit also satisfies the same stable excision condition. This follows because homotopy groups commute with directed colimits; that is, directed homotopy colimits preserve injections and surjections on homotopy groups, and hence k -connected maps. \square

Recall that $D_n F$ is the homogeneous n -excisive functor that is the homotopy fiber of the map from the n -excisive approximation $P_n F$ to the $(n - 1)$ -excisive approximation $P_{n-1} F$.

Lemma 6.6. *If F satisfies $E_n(c)$, then $D_{n+1} F$ satisfies $E_n(c)$ as well.*

Proof. We first show that we can reduce to considering strongly co-Cartesian cubes with contractible initial object. This type of cube is k -Cartesian if the

(homotopy inverse limit of the) punctured cube is k -connected, so we can determine Cartesian-ness by connectivity of a space. We then commute homotopy inverse limits to compute the connectivity of the punctured D_{n+1} cube from that of P_{n+1} and P_n .

Consider the fibration sequence

$$D_{n+1}F \rightarrow P_{n+1}F \rightarrow P_nF.$$

By Corollary 6.5, the total space $P_{n+1}F$ satisfies $E_n(c)$, and of course P_nF is n -excisive.

Let T be a set of cardinality $n + 1$, and let \mathcal{X} be a strongly co-Cartesian T -cube. Define the strongly co-Cartesian T -cube \mathcal{Y} by coning off the initial vertex $\mathcal{X}(\emptyset)$ of \mathcal{X} :

$$\mathcal{Y}(U) = \operatorname{colim}(C(\mathcal{X}(\emptyset)) \leftarrow \mathcal{X}(\emptyset) \rightarrow \mathcal{X}(U))$$

Since the functors D_{n+1} , P_{n+1} and P_n are all $(n + 1)$ -excisive, they each take the $(n + 2)$ -cube $\mathcal{X} \rightarrow \mathcal{Y}$ to a Cartesian cube. So after applying any one of these functors, if the functored sub-cube \mathcal{Y} is k -Cartesian then the functored sub-cube \mathcal{X} is k -Cartesian (by [18, Proposition 1.6, p. 303]). The sub-cube \mathcal{Y} has contractible initial vertex, which is the case we wanted to reduce to. Now assume, using this reduction if necessary, that \mathcal{X} is a strongly co-Cartesian T -cube with contractible initial vertex. Since $D_{n+1}F$ is reduced and $\mathcal{X}(\emptyset) \simeq *$, the connectivity of the map

$$D_{n+1}F\mathcal{X}(\emptyset) \rightarrow \operatorname{holim}_{U \in \mathcal{P}_0(T)} D_{n+1}F\mathcal{X}(U)$$

is determined by the connectivity of $\operatorname{holim}_{U \in \mathcal{P}_0(T)} D_{n+1}F\mathcal{X}(U)$ (since $D_{n+1}F\mathcal{X}(\emptyset) \simeq *$). If our functor is not reduced, there is a fibration over $F(*)$ with fiber a reduced functor, so there is no real difference in the arguments in this case; they are just made relative to $F(*)$.

We can then compute:

$$\begin{aligned} \operatorname{holim}_{U \in \mathcal{P}_0(T)} D_{n+1}F\mathcal{X}(U) &= \operatorname{holim}_{U \in \mathcal{P}_0(T)} \operatorname{hofib}(P_{n+1}F\mathcal{X}(U) \rightarrow P_nF\mathcal{X}(U)) \\ &\simeq \operatorname{hofib}\left(\operatorname{holim}_{U \in \mathcal{P}_0(T)} P_{n+1}F\mathcal{X}(U) \rightarrow \operatorname{holim}_{U \in \mathcal{P}_0(T)} P_nF\mathcal{X}(U)\right) \end{aligned}$$

The n -excisiveness of P_nF implies that the inverse limit of the punctured P_n cube is equivalent to $P_nF\mathcal{X}(\emptyset) \simeq F(*)$, and the inverse limit of the punctured P_{n+1} cube has connectivity at least $\sum k_i - c$ relative to $F(*)$ because $P_{n+1}F$ satisfies $E_n(c)$, so the homotopy fiber also has connectivity at least $\sum k_i - c$. Hence $D_{n+1}F$ satisfies $E_n(c)$ as well. \square

Lemma 6.7. *Let \mathbf{C} be a spectrum with a Σ_{n+1} action, and let F be the functor*

from spaces to spectra given by

$$F(X) = \mathbf{C} \wedge_{h\Sigma_{n+1}} X^{\wedge(n+1)},$$

so F is homogeneous of degree $n + 1$, with $n \geq 0$. If F satisfies $E_n(c)$, then $\pi_j \mathbf{C} = 0$ for $j < -c$.

Proof. We will show that \mathbf{C} cannot have nonzero homotopy groups in dimensions lower than is claimed by showing that if so, the result would be a functor that does not satisfy $E_n(c)$. The condition $E_n(c)$ gives information about the Cartesian-ness of F applied to certain cubes, so we will compute a bound on the Cartesian-ness by computing the connectivity of the total fiber.

Recall that in the category of spectra, the total cofiber and total fiber are related by a shift in dimension equal to the dimension of the cube. To compute the total fiber of our functor

$$F(X) = \operatorname{hocolim}_{\Sigma_{n+1}} (\mathbf{C} \wedge X^{\wedge(n+1)})$$

applied to the cube, we first compute the total cofiber of the $(n + 1)^{\text{st}}$ smash power, then smash with \mathbf{C} , then take Σ_{n+1} orbits, and finally loop back n times for the dimension shift. That is, we compute

$$\begin{aligned} \operatorname{hocolim}_{\Sigma_{n+1}} \left(\mathbf{C} \wedge \operatorname{total\,cofib} \left(X^{\wedge(n+1)} \right) \right) &\simeq \operatorname{hocolim}_{\Sigma_{n+1}} \left(\operatorname{total\,cofib}_{X \in \mathcal{X}} \left(\mathbf{C} \wedge X^{\wedge(n+1)} \right) \right) \\ &\simeq \operatorname{total\,cofib}_{X \in \mathcal{X}} \left(\operatorname{hocolim}_{\Sigma_{n+1}} \left(\mathbf{C} \wedge X^{\wedge(n+1)} \right) \right) \\ &\simeq \operatorname{total\,cofib}_{X \in \mathcal{X}} F(X). \end{aligned}$$

The second equivalence is because both the total cofiber and the homotopy orbits are colimit constructions, and hence commute. This shows that the cofiber we compute is actually that of F applied to the cube.

Let X be a space and consider the strongly co-Cartesian $(n + 1)$ -cube \mathcal{X} generated by $\mathcal{X}(\emptyset) = *$ and $\mathcal{X}(\{i\}) = X$, so $\mathcal{X}(U) = \bigvee_U X$. Let \mathcal{Y} be the cube $\mathcal{X}^{\wedge(n+1)}$, with $\mathcal{Y}(U) = \mathcal{X}(U)^{\wedge(n+1)}$.

The total cofiber of the cube \mathcal{Y} is equivalent to the $n + 1$ cross effect of the $n + 1$ smash power, $cr_{n+1}(\bigwedge^{n+1})$. Writing $X_i \cong X$ to make the action of Σ_{n+1} clear, this is:

$$cr_{n+1} \left(\bigwedge^{n+1} \right) (X_1, \dots, X_{n+1}) \xleftarrow{\simeq} \bigvee_{\sigma \in \Sigma_{n+1}} X_{\sigma(1)} \wedge \dots \wedge X_{\sigma(n+1)}.$$

Fortunately, it is easy to see that the right hand side is a free Σ_{n+1} space, so smashing with \mathbf{C} and taking homotopy orbits gives $\mathbf{C} \wedge X^{\wedge(n+1)}$. Hence for this cube, the total fiber is $\Omega^n(\mathbf{C} \wedge X^{\wedge(n+1)})$.

The Cartesian-ness of the cube $F\mathcal{X}$ is determined by the connectivity of the

total fiber. When the space X is m -connected then $X^{\wedge(n+1)}$ is $((n+1)(m+1) - 1) = ((n+1)m + n)$ -connected. If \mathbf{C} has its bottom nonzero homotopy group in dimension w , the total fiber has connectivity $(n+1)m + n - n + w = (n+1)m + w$. Since F satisfies $E_n(c)$, we must have $w \geq -c$. \square

Corollary 6.8. *Let \mathbf{C} be a spectrum with a Σ_{n+1} action, and let F be the functor from spaces to spaces given by*

$$F(X) = \Omega^\infty(\mathbf{C} \wedge_{h\Sigma_{n+1}} X^{\wedge(n+1)}),$$

so F is homogeneous of degree $n+1$, with $n \geq 0$. If F satisfies $E_n(c)$, then $\pi_j \mathbf{C} = 0$ for $j < -c$.

Proof. Choosing X to be highly enough connected, all of the maps in the cube of Lemma 6.7 are connected enough that the Cartesian-ness of the cube is determined by the connectivity of the total fiber. Again, if X is connected enough, the total fiber will be connective, so after the application of Ω^∞ (which preserves fibers), it will have the same connectivity as the fiber as spectra, so we the result for spaces follows. \square

Proof of Theorem 6.1. Without loss of generality, we may assume that F is reduced. We do this by replacing F with the functor $\tilde{F}(X) = \text{hofib}(F(X) \rightarrow F(0))$ and observing that they satisfy the same excision conditions $E_n(c)$ for all $n \geq 0$. If F is r -analytic, then F satisfies $E_n(rn - c)$, so by Lemma 6.6, $D_{n+1}F$ satisfies $E_n(rn - c)$, so by Corollary 6.8, $\pi_j \mathbf{C}_{n+1} = 0$ for $j < c - rn$. \square

6.2 Ω^∞ Commutes With Certain Realizations

The result of this section is that when all of the spectra X_i in a simplicial spectrum X are connective, the functor Ω^∞ can be applied before or after realization, with the same results. (Recall that we call a spectrum connective if all of its negative homotopy groups are zero.)

Theorem 6.9. *If X is a simplicial connective spectrum, then the simplicial spectrum $\Omega^\infty ||[n] \mapsto X_n||$ is equivalent to the simplicial spectrum $||[n] \mapsto \Omega^\infty X_n||$.*

The remainder of this section consists of the proof of this theorem and subsidiary results required therein. Let S^1 be the standard model $\Delta^1 / \partial \Delta^1$ (where Δ^1 is $[n] \mapsto \text{Hom}([n], [1])$) for the simplicial 1-sphere.

Lemma 6.10. *If X is a connective spectrum, then*

$$||[n] \mapsto \Omega^\infty(S_n^1 \wedge X)||$$

is equivalent to

$$\Omega^\infty ||[n] \mapsto S_n^1 \wedge X||.$$

Furthermore, both have the same infinite loop space structure.

A proof of this lemma different from the one that follows appears in the literature in [2].

Proof. Consider the levelwise cofiber sequence of simplicial sets $S^0 \rightarrow D^1 \rightarrow S^1$. Applying the two functors in question, we have the following diagram:

$$\begin{array}{ccccc} ||\Omega^\infty(S^0 \wedge X)|| & \longrightarrow & ||\Omega^\infty(D^1 \wedge X)|| & \longrightarrow & ||\Omega^\infty(S^1 \wedge X)|| \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^\infty||S^0 \wedge X|| & \longrightarrow & \Omega^\infty||D^1 \wedge X|| & \longrightarrow & \Omega^\infty||S^1 \wedge X|| \end{array}$$

Note that $S^0 \wedge X$ is a trivial simplicial set, so the leftmost map is an equivalence. Also, D^1 is simplicially contractible, so both of the spaces appearing in the middle are contractible (and hence the map is an equivalence). The bottom row is a fibration (up to homotopy) because cofiber sequences of spectra are equivalent to fiber sequences, and Ω^∞ preserves fiber sequences.

The top row is also a fiber sequence. This depends on the fact that $\Omega^\infty(S^1 \wedge X)$ is the simplicial bar construction on $\Omega^\infty X$, and that this produces a delooping of $\Omega^\infty X$ (by [23, Proposition 1.4, p. 295]). Using the stated model for S^1 , we have $S^1_n \wedge X = [n] \wedge X = \bigvee_n X \simeq \prod_n X$, since finite coproducts and products are equivalent in spectra. Applying Ω^∞ (which commutes with products), we have the simplicial object $[n] \mapsto (\Omega^\infty X)^{\times n}$. We leave skeptical readers to verify for themselves that the fold map $X \vee X \rightarrow X$ of spectra induces the product map for the H -space $\Omega^\infty X$ (via the equivalence $X \times X \simeq X \vee X$). Using the five lemma, we immediately find that the right hand map is an equivalence on π_i , for $i \geq 1$. The space $||\Omega^\infty(S^1 \wedge X)||$ is always connected, because there are no zero simplices. Since X is connective, we also have $\Omega^\infty||S^1 \wedge X||$ connected, so $\pi_0 = 0$ in both cases. This shows that the right-hand map is a weak equivalence. \square

Corollary 6.11. *If X is a connective spectrum, then $||\Omega^\infty(S^n \wedge X)|| \simeq \Omega^\infty||S^n \wedge X||$.*

Proof. Write S^n as the diagonal of the bisimplicial set $S^1 \wedge S^{n-1}$. Use the Eilenberg-Zilber theorem to replace with the whole bisimplicial set, and apply Lemma 6.10 inductively. \square

We will demonstrate that all grouplike H -spaces satisfy the π_* -Kan condition, giving us a large class of examples.

Definition 6.12 (Simple space). A connected space X is called *simple* if $\pi_1 X$ is abelian and acts trivially on the higher homotopy groups. A general space X is called *simple* if each component of X is a simple space.

The following lemma appears as an exercise in [8].

Lemma 6.13. ([8, B.3.1, p. 120]) *Let X be a simplicial space, and let $[S^t, -]$ denote the unpointed homotopy classes of (unpointed) maps out of S^t . If each*

X_m is a simple space, then

$$[S^t, X.] \rightarrow \pi_0 X. \quad (6.14)$$

is a fibration of simplicial sets if and only if $X.$ satisfies the π_* -Kan condition.

□

Proof. Recall from 2.11 that the π_* -Kan means that if given $t \geq 1$ and $a \in X_{m+1}$ and a coherent collection $x_i \in \pi_t(X_m, \partial_i a)$, with $0 \leq i \leq m+1$ and $i \neq k$, there exists a $y \in \pi_t(X_{m+1}, a)$ with $\partial_i y = x_i$ for $i \neq k$. Also, recall that map $p : E \rightarrow B$ of simplicial sets is a fibration if given $a \in B_{m+1}$ and a coherent collection $x_i \in E_m$ with $p(x_i) = \partial_i a$, for $0 \leq i \leq m+1$ and $i \neq k$, there exists a $y \in E_{m+1}$ with $p(y) = a$ and $\partial_i y = x_i$ for $i \neq k$.

In a simple space, elements of $\pi_t(X_m, \partial_i a)$ are in bijective correspondence with free homotopy classes of maps of S^t to X_m that land in the component of $\partial_i a$. (In general, allowing free homotopies identifies maps that are the same orbit under the action of π_1 .) Now comparing the definitions of a fibration of simplicial sets and the π_* -Kan condition to verify that if X satisfies the π_* -Kan condition, then (6.14) is a fibration.

If (6.14) is a fibration, then we can use this fact to produce a $[y] \in [S^t, X_{m+1}]$ that lands in the same path component as a , and satisfying $\partial_i [y] = [x_i]$ for $i \neq k$. This $[y]$ can be realized as a map $y : S^t \rightarrow X_{m+1}$ that takes the basepoint of S^t to a . Now $\partial_i y : (S^t, *) \rightarrow (X_m, \partial_i a)$, and $[\partial_i y] = [x_i]$. But the free homotopy classes of maps $[x_i]$ landing in the path component of $\partial_i a$ are in one-to-one correspondence with the elements of $\pi_t(X_m, \partial_i a)$, so $\partial_i y$ must actually be x_i . This shows that $X.$ satisfies the π_* -Kan condition. □

Corollary 6.15. *If $X.$ is a simplicial grouplike H -space, then $X.$ satisfies the π_* -Kan condition.*

Proof. All H -spaces are simple, so Lemma 6.13 can be used. The map $[S^t, X.] \rightarrow \pi_0 X.$ is obviously surjective. We will show that both the source and target are simplicial groups; all surjections of simplicial groups are fibrations, so this will allow us to apply Lemma 6.13 to conclude that $X.$ satisfies the π_* -Kan condition.

Since $X.$ is a grouplike H -space, the simplicial set $\pi_0 X.$ is actually a simplicial group.

The set $[S^t, X_m]$ is a group with multiplication induced by the H -space multiplication on X_m . A grouplike H -space only satisfies the axioms for a group up to homotopy, but we are considering homotopy classes of maps, so that is not a problem. □

Corollary 6.16. *Let $X.$ be a simplicial space. The simplicial space $[n] \mapsto \Omega X_n$ satisfies the π_* -Kan condition. In particular, simplicial infinite loop spaces (arising from $[n] \mapsto \Omega^\infty X_n$) satisfy this condition.*

Proof. Loop spaces are grouplike H -spaces, so Corollary 6.15 applies. □

We are finally ready to begin the proof of the main result of this section.

Proof of Theorem 6.9. Let X_\bullet be a simplicial connective spectrum. If necessary, begin by functorially replacing X_\bullet by a weakly equivalent simplicial connective spectrum in which the degeneracy maps are cofibrations. Let $\overline{X_n}$ denote the quotient of X_n by the union of the images of the degeneracy maps. The degeneracy maps are cofibrations, so the strict quotient is a homotopy invariant. Since we are in the category of spectra (a stable category), we can split off the degenerate elements up to equivalence, giving the standard decomposition:

$$X_n \simeq \bigvee_{\text{Surj}(n,k)} \overline{X_k},$$

where $\text{Surj}(n, k)$ denotes the surjective maps from $[n]$ to $[k]$ in Δ . Each degeneracy map s_j has an inverse d_j , so if each X_k is connective, then so is each $\overline{X_k}$. This decomposition lets us identify the cokernel of the inclusion of the $(n-1)$ skeleton into the n skeleton.

We now proceed by induction up the simplicial skeleta. Let $X_{\leq n}$ denote the simplicial n -skeleton of X_\bullet . The inclusion of the $(n-1)$ -skeleton into the n -skeleton gives rise to a (levelwise) cofibration sequence

$$X_{\leq n-1} \rightarrow X_{\leq n} \rightarrow S^n \wedge \overline{X_n}. \quad (6.17)$$

Since cofibration sequences and fibration sequences are equivalent for spectra, and Ω^∞ preserves fibration sequences,

$$\Omega^\infty X_{\leq n-1} \rightarrow \Omega^\infty X_{\leq n} \rightarrow \Omega^\infty(S^n \wedge \overline{X_n})$$

is a levelwise fibration sequence of spaces. Since $\overline{X_n}$ is connective, and $X_{\leq n-1}$ is connective by induction, the fibration is necessarily surjective (levelwise) on π_0 (because the long exact sequence of a fibration sequence of spectra continues past π_0 to π_{-1} , which is 0 in this case). Furthermore, since π_0 of an infinite loop space is a group, this is actually a surjective map of simplicial groups, which is fortunately a fibration ([26, Exercise 8.2.5, p. 262]). By Corollary 6.16, both $\Omega^\infty X_{\leq n}$ and $\Omega^\infty(S^n \wedge \overline{X_n})$ satisfy the π_* -Kan condition. A theorem of Bousfield and Friedlander (Theorem 2.12) now shows that we have a fibration after realization as well:

$$\|\Omega^\infty X_{\leq n-1}\| \rightarrow \|\Omega^\infty X_{\leq n}\| \rightarrow \|\Omega^\infty(S^n \wedge \overline{X_n})\|.$$

The realization of (6.17) is still a cofibration sequence, and hence a fibration sequence, so we also have a fibration sequence

$$\Omega^\infty \|X_{\leq n-1}\| \rightarrow \Omega^\infty \|X_{\leq n}\| \rightarrow \Omega^\infty \|S^n \wedge \overline{X_n}\|.$$

Combining these two fibration sequences gives rise to the commutative dia-

gram:

$$\begin{array}{ccccc}
||\Omega^\infty X_{\leq n-1}|| & \longrightarrow & ||\Omega^\infty X_{\leq n}|| & \longrightarrow & ||\Omega^\infty(S^n \wedge \overline{X_n})|| \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^\infty ||X_{\leq n-1}|| & \longrightarrow & \Omega^\infty ||X_{\leq n}|| & \longrightarrow & \Omega^\infty ||S^n \wedge \overline{X_n}||
\end{array}$$

The map on the fibers is an equivalence by induction, and the map on the bases is an equivalence by Lemma 6.10, so the map on the total spaces is certainly an isomorphism on π_j for $j \geq 1$ (using the five lemma). Actually, all of the π_0 are abelian groups, so the five lemma implies the total spaces are equivalent. This is obvious for the bottom row, since $\pi_0 \Omega^\infty(X) = \pi_0 X$ is an abelian group. In the top row, we use the fact that $\pi_0 ||\Omega^\infty Y||$ is the quotient of $\pi_0 \Omega^\infty Y_0$ by $\pi_0 \Omega^\infty Y_1$, and the map $\Omega^\infty Y_0 \rightarrow \Omega^\infty Y_1$ is an infinite loop map (and hence a map of groups). \square

Corollary 6.18. *Let \mathbf{C} be a spectrum, and let $F(X) = \Omega^\infty(\mathbf{C} \wedge X^{\wedge n})_{h\Sigma_n}$ be a (homogeneous) functor from spaces to spaces. If X is a simplicial space such that $\mathbf{C} \wedge X_i^{\wedge n}$ is connective for all i , then $F(||X.||) \simeq ||F(X.)||$; that is, F commutes with the realization of X .*

Proof. Corollary 5.11 shows that functors of finite degree from spaces to spectra commute with realizations. Proposition 6.9 shows that Ω^∞ commutes with realizations of all simplicial connective spectra. The spectrum $\mathbf{C} \wedge X_i^{\wedge n}$ is connective, and taking homotopy orbits does not lower connectivity, so the result is an immediate corollary of combining those two. \square

6.3 Analytic Functors Commute With Highly Connected Realizations

This section uses the results of the previous two sections (§6.1 and §6.2) to show that analytic functors with the limit axiom commute with realizations of simplicial k -connected spaces, for sufficiently large k .

Theorem 6.19. *Let F be a reduced analytic functor from spaces to spaces satisfying the limit axiom (5.1) and the stable excision condition $E_n(rn - c)$ for all n (as defined in §3.3). If X is a simplicial k -connected space, with $k \geq \max(r, -c)$, then $F(||X.||) \simeq ||F(X.)||$. That is, F commutes with realizations of simplicial k -connected spaces.*

Proof. Recall that smashing with a space of connectivity k increases connectivity by $(k + 1)$. By Theorem 6.1, the coefficient spectrum \mathbf{C}_{m+1} , for $m \geq 0$, has its bottom nonzero homotopy group in dimension $(c - rm)$. Computing $\mathbf{C}_{m+1} \wedge$

$X_i^{\wedge(m+1)}$, we find that its bottom nonzero homotopy group is in dimension

$$\begin{aligned}(c - rm) + (m + 1)(k + 1) &= (k + c) + (k - r)m + (m + 1) \\ &\geq (0) + (0)m + m + 1,\end{aligned}$$

so in particular it is always connected. Therefore, we may apply Corollary 6.18 to conclude that $D_n F$ commutes with the realization of X .

We now induct up the Taylor tower to show that each $P_n F$ commutes with the realization of X . To start the induction, note that $P_1 F = D_1 F$ is connected and commutes with the realization of X . Then suppose inductively that $P_n F$ is connected and commutes with the realization of X . Theorem 3.14 says that $P_{n+1} F(X)$ can be computed as the homotopy fiber of a map $P_n F(X) \rightarrow \Omega^{-1} D_{n+1} F(X)$. Our connectivity estimate from the previous paragraph shows that under our hypotheses, $D_{n+1} F$ is simply connected for $n \geq 1$ when evaluated on each X_i , and commutes with the realization of X . Since $D_{n+1} F(X_i)$ is connected, we may apply Lemma 2.10 to compute $P_{n+1} F(\|X\|)$ as $\|P_{n+1} F(X)\|$, so $P_{n+1} F$ also commutes with the realization of X .

The connectivity of the map $F(X) \rightarrow P_n F(X)$ grows with n and the connectivity of X , provided that the connectivity of X is at least r ; that is, X is within the radius of convergence. This is the case under our hypotheses, and each $P_n F$ commutes with the realization of X , so F must also commute with the realization of X . \square

Chapter 7

Cotriples For Additive Functors

The most basic notion of “degree” of a functor is something that is called “additive degree”. A functor is additive degree n if $F(\bigvee^n X)$ is determined by $F(\bigvee^k X)$ for $k < n$, in the sense that $F(\bigvee^n X)$ is the inverse limit of a certain diagram involving only the $F(\bigvee^k X)$, for $k < n$. The algebraic intuition for additive degree n , or “ n -additive”, functors has the same roots as for n -excisive functors: a polynomial of degree n is determined by its values on $n + 1$ points; that is, the set $\{\bigvee^k S^0 \mid 0 \leq k \leq n\}$. The difference between additive and excisive functors is that if F is only additive, there need not be any relationship between $F(X)$ and $\Omega F(\Sigma X)$. Our interest in additive functors stems from the fact that in many cases the difference between F and its additive approximation results from a standard construction called a “cotriple”. Use of this construction provides a spectral sequence to compute the homotopy groups of the n -additive approximation to a functor, and plays an important role in our understanding of n -additive functors in general.

7.1 Additivity, Homotopy Fibers, And Special Notation

In this section, we make precise what we mean by an n -additive functor.

Throughout this chapter, we will mainly be concerned with cubes made up of coproducts of spaces X_α . Let T be a set, let $\{X_\alpha\}_{\alpha \in T}$ be a collection of spaces, and define the T -cube \mathcal{X} by

$$\mathcal{X}_T^{\{X_\alpha\}}(U) = \bigvee_{\alpha \in T-U} X_\alpha$$

with the inclusion $i : U \hookrightarrow V$ inducing the map $\mathcal{X}(i)$ given by

$$\mathcal{X}(i)(X_\alpha) = \begin{cases} * & \alpha \in V \\ X_\alpha & \alpha \notin V \end{cases}$$

When all of the X_α are the same space X , we will write \mathcal{X}_T^X for the cube. For notational convenience, we will immediately suppress the dependence of \mathcal{X} on X_α and T unless it is not clear from context. Notice that such a cube \mathcal{X} is equivalent to strongly co-Cartesian cube, since one could include X_α in the cone over X_α instead of collapsing it to a single point. The equivalent cube would then be:

$$\mathcal{X}_T^{\{X_\alpha\}}(U) = \bigvee_{\alpha \in T-U} X_\alpha \vee \bigvee_{\alpha \in U} CX_\alpha.$$

Definition 7.1 (n -additive). A functor F is n -additive if the $(n+1)$ -cube $F\mathcal{X}_{n+1}^X$ is Cartesian for all spaces X .

Remark 7.2. We do not require that $F\mathcal{X}_T^{\{X_\alpha\}}$ be Cartesian for arbitrary collections of spaces X_α ; only those with all X_α the same space X .

We can rephrase the definition of n -additivity as follows: for all X , an n -additive functor F gives an equivalence

$$F\mathcal{X}_{n+1}^X(\emptyset) \xrightarrow{\cong} \operatorname{holim}_{U \in \mathcal{P}_0(S)} F\mathcal{X}_{n+1}^X(U).$$

Our approach to n -additivity will be to break the problem of understanding this map into two parts: we show when the (homotopy) fiber of this map is contractible, and understand some general conditions under which the map is surjective on π_0 . These two parts combined allow us to understand when $F\mathcal{X}_{n+1}^X$ is Cartesian. We use the term *homotopy fiber* of a cube to describe the homotopy fiber of a map like the one above.

Definition 7.3 (Homotopy fiber). ([18, 1.1]) Let \mathcal{X} be an S -cube of pointed spaces, and for $T \subset S$, define the topological cube I^T to be the product of T copies of the unit interval $I = [0, 1]$. (When $T = \emptyset$, this is interpreted as $I^\emptyset = \{0\}$.) A point $\Phi \in \operatorname{hofib} \mathcal{X}$ is a collection of continuous maps $\Phi_T : I^T \rightarrow \mathcal{X}(T)$, one for each subset $T \subset S$, satisfying the two conditions below.

1. Φ is natural with respect to T . That is, for $U \subset T \subset S$, the following diagram commutes:

$$\begin{array}{ccc} I^U & \xrightarrow{\quad} & I^T \\ \downarrow \Phi_U & & \downarrow \Phi_T \\ \mathcal{X}(U) & \longrightarrow & \mathcal{X}(T) \end{array}$$

where the upper arrow is the map that takes a function $U \rightarrow I$ and extends it to a function $T \rightarrow I$ by making it zero on $T - U$.

2. For each $T \subset S$, the function Φ_T takes the set of points with at least one coordinate having value one, $(I^T)_1 := \{u \in I^T : \exists_{s \in T} u_s = 1\}$, to the basepoint in $\mathcal{X}(T)$.

This definition of the homotopy fiber of a cube is homeomorphic to defining the homotopy fiber of an S -cube \mathcal{X} to be the homotopy fiber of the map $\mathcal{X}(\emptyset) \rightarrow \text{holim}_{U \in \mathcal{P}_0(S)} \mathcal{X}(U)$. It also agrees with the construction of the homotopy fiber given inductively by repeatedly taking homotopy fibers of the structure maps in a single direction.

7.2 Cross Effects

The n -th cross effect of a functor is a functorial comparison of $F(\bigvee^n X)$ with F on lower order coproducts of X . By taking the homotopy fiber of maps to smaller coproducts of X , the cross effect “kills off” their contribution to $F(\bigvee^n X)$, leaving only the part that does not “come from” lower order coproducts of X .

Definition 7.4 (cr_n, \perp_n). Define the n^{th} cross effect of a functor F to be the functor of n variables

$$cr_n F(X_1, \dots, X_n) = \text{hofib}_{U \in \mathcal{P}(\mathbf{n})} F\mathcal{X}_{\mathbf{n}}^{\{X_i\}}(U),$$

where \mathbf{n} denotes the set $\{1, \dots, n\}$. In this notation, the subscripts of the X_α on the left correspond to the X_α in \mathcal{X} on the right.

Let $\perp_n F(X) = cr_n F(X, \dots, X)$ be the diagonal of the n^{th} cross effect of F evaluated at X . Denote the iteration of this functor by $\perp_n^{(a)} F$. We will later show that \perp_n is part of a cotriple.

Abuse of notation: At some points in Section 9.2, we need to discuss $\perp_n F$ as a functor of n variables. At those points, we will write $\perp_n F(X, \dots, X)$, understanding that this is the same as $cr_n F(X, \dots, X)$, and hope that this causes no confusion.

Actually, the vanishing of the cross effect cr_n for all choices of inputs is equivalent to the vanishing of \perp_n .

Lemma 7.5. *The functor $cr_n F(X_1, \dots, X_n)$ is contractible for all choices of inputs X_i if and only if $\perp_n F(X)$ is contractible for all X .*

Proof. Since $\perp_n F(X) = cr_n F(X, \dots, X)$, one implication is trivial. Now suppose $\perp_n F(X)$ is contractible for all X . Given $\{X_i\}_{i=1}^n$, let $X = \bigvee X_i$. Let $i_j : X_i \rightarrow X$ denote the inclusion of the j^{th} factor, and let p_j denote the projection onto the j^{th} factor. The map

$$cr_n F(p_1, \dots, p_n) : \perp_n F(X) = cr_n F(X, \dots, X) \rightarrow cr_n F(X_1, \dots, X_n)$$

has a section $cr_n F(i_1, \dots, i_n)$, so if $\perp_n F(X) \simeq *$, then $cr_n F(X_1, \dots, X_n) \simeq *$ (e.g., because $\pi_*(Id)$ factors through 0). \square

Since homotopy inverse limits commute, $\perp_n^{(a)} F(X)$ may be computed by

$$\perp_n^{(a)} F = \operatorname{hofib}_{U \in \mathcal{P}(\mathbf{n})^a} F\mathcal{X}(U).$$

Example 7.6. The second iterated cross effect, $\perp_n^2 F(X)$, is

$$\begin{aligned} \perp_n^{(2)} F(X) &= \operatorname{hofib}_{V_2 \in \mathcal{P}(\mathbf{n})} (\perp_n F) \mathcal{X}_{\mathbf{n}}^X(V_2) \\ &= \operatorname{hofib}_{V_2 \in \mathcal{P}(\mathbf{n})} \left(\operatorname{hofib}_{V_1 \in \mathcal{P}(\mathbf{n})} F\mathcal{X}_{\mathbf{n}}^{X(V_2)}(V_1) \right) \\ &= \operatorname{hofib}_{\substack{V_1 \in \mathcal{P}(\mathbf{n}) \\ V_2 \in \mathcal{P}(\mathbf{n})}} F\mathcal{X}_{\mathbf{n}}^{X(V_2)}(V_1) \end{aligned}$$

To decode the cube \mathcal{X} that appears, recall that the superscript denotes the space from which the coproducts are formed, so we have:

$$\begin{aligned} \mathcal{X}_{\mathbf{n}}^{X(V_2)}(V_1) &= \bigvee_{v_1 \notin V_1} \mathcal{X}_{\mathbf{n}}^X(V_2) \\ &= \bigvee_{v_1 \notin V_1} \bigvee_{v_2 \notin V_2} X \end{aligned}$$

From this example, the general form of the cubes used to compute $\perp^{(a)}$ for $a > 2$ should be clear.

In order to work with cross effects, we need to establish certain basic properties. One of the most fundamental is that all cross effects can be built up by iterating the second cross effect.

In the next few results, we consider cross-effect cubes as functorial in the spaces that are used to create them, and write $\mathcal{X}[X_1, \dots]$ to indicate the cube \mathcal{X} built using the spaces X_1 , *etc.*

Recall that the \mathbf{n} -cube defining $cr_n F(Y_1, \dots, Y_n)$ is given by

$$\mathcal{Y}[Y_1, \dots, Y_n](U) = F \left(\bigvee_{b \notin U} Y_b \right).$$

Consider the result of applying the functor $cr_2(-)(X_1, X_2)$ in the first variable of this functor. That gives the $\mathbf{2}$ -cube of \mathbf{n} -cubes:

$$\mathcal{Z}[X_1, X_2](V) = \mathcal{Y}(U) \left[\bigvee_{v \notin V} X_v, Y_2, \dots, Y_n \right].$$

Now for brevity, let $S = \mathbf{n} \amalg \mathbf{2}$, and let $T = S - \{1\} \amalg \emptyset$. The cube \mathcal{Z} can be written as an S -cube by defining $X_{k+1} = Y_k$, for $k = 2, \dots, n$, so the cube is:

$$\mathcal{W}[X_1, \dots, X_{n+1}](U \amalg V) = \begin{cases} F \left(\bigvee_{v \notin V} X_v \vee \bigvee_{u \notin U \cup \{1\}} X_{u+1} \right) & 1 \notin U \\ F \left(\bigvee_{u \notin U} X_{u+1} \right) & 1 \in U \end{cases}$$

We will immediately suppress writing the $[X_1, \dots, X_{n+1}]$ except where the spaces X_i are relevant.

There are two things to note. First, the cube used to compute $cr_{n+1}F(X_1, \dots, X_{n+1})$ is exactly the $(n+1)$ -cube

$$\{A \mapsto \mathcal{W}[X_1, \dots, X_{n+1}](A) : A \subset T\},$$

which Goodwillie denotes $\partial^T \mathcal{W}[X_1, \dots, X_{n+1}]$. Second, when $1 \in U$, the subcube $\mathcal{W}(U \amalg -)$ is a constant cube, so the other $(n+1)$ -cube, $\partial_{\{1\} \amalg \emptyset} \mathcal{W}$, that makes up \mathcal{W} consists of a cube of constant 2-cubes.

Lemma 7.7. *If the $(n+2)$ -cube used to compute the second cross effect of $cr_n F(Y_1, \dots, Y_n)$ in a single variable, e.g., Y_1 , is Cartesian, then the $(n+1)$ -cube defining $cr_{n+1}F(X_1, \dots, X_{n+1})$ is Cartesian.*

Proof. As discussed above, the cube $\mathcal{W}[X_1, \dots, X_{n+1}]$ is the $(n+2)$ -cube that is used to compute the second cross effect with respect to X_1 and X_2 of the functor $cr_n F(-, X_3, \dots, X_{n+1})$, and $\partial^T \mathcal{W}[X_1, \dots, X_{n+1}]$ is the $(n+1)$ -cube used to compute $cr_n F(X_1, \dots, X_{n+1})$, so we need to establish that \mathcal{W} is Cartesian if and only if $\partial^T \mathcal{W}$ is Cartesian. We will do this by showing that the bottom arrow on the following commutative diagram is an equivalence, and hence if either one of the two vertical maps is an equivalence, then so is the other.

$$\begin{array}{ccc} \mathcal{W}(\emptyset) & & \\ \downarrow & \searrow & \\ \operatorname{holim}_{\mathcal{P}_0(S)} \mathcal{W} & \longrightarrow & \operatorname{holim}_{\mathcal{P}_0(T)} \mathcal{W} \end{array}$$

We will compare the homotopy inverse limits of \mathcal{W} over $\mathcal{P}_0(S)$ and $\mathcal{P}_0(T)$ in two stages. Let $\mathcal{R} = \mathcal{P}_0(S) - \{1\} \amalg \emptyset$ be the category $\mathcal{P}_0(S)$ without the object $\{1\} \amalg \emptyset$. The inclusions $\mathcal{P}_0 T \hookrightarrow \mathcal{R} \hookrightarrow \mathcal{P}_0 S$ induce maps

$$\operatorname{holim}_{\mathcal{P}_0(S)} \mathcal{W} \rightarrow \operatorname{holim}_{\mathcal{R}} \mathcal{W} \rightarrow \operatorname{holim}_{\mathcal{P}_0(T)} \mathcal{W}. \quad (7.8)$$

We will show that both of these maps are equivalences. The right hand map in (7.8) is an equivalence by [9, §XI.9, Theorem 9.2] because $\mathcal{P}_0 T$ is left cofinal in \mathcal{R} . To verify left cofinality, let $U \amalg V \in \operatorname{Obj}(\mathcal{R})$ (that is, $\{1\} \amalg \emptyset \neq U \amalg V \subset S$). The set $(U - \{1\}) \amalg V$ is in $\mathcal{P}_0(T)$ (because the restriction on $U \amalg V$ guarantees that this is not $\emptyset \amalg \emptyset$), so the category $(\mathcal{P}_0(T) \rightarrow \mathcal{R})/U \amalg V$ contains the object $(U - \{1\}) \amalg V$ with the inclusion map $(U - \{1\}) \amalg V \rightarrow U \amalg V$. Hence this category is nonempty. Morphisms in $\mathcal{P}_0(-)$ are inclusions of subsets, so there is at most one morphism between any two objects; hence $U \amalg V \rightarrow U \amalg V$ is the terminal object in this category, and it is contractible. The left hand map in (7.8) is an equivalence for reasons particular to the cube \mathcal{W} , as we will now show. Recall that a homotopy inverse limit of a functor \mathcal{W} over a category \mathcal{D}

is the space of maps $\text{Map}_{\mathcal{D}}(\|\mathcal{D}/-\|, \mathcal{W}(-))$. Let $i : U \rightarrow V$ be a morphism in \mathcal{D} , and let $\phi \in \text{Map}_{\mathcal{D}}(\|\mathcal{D}/-\|, \mathcal{W}(-))$. Then the components ϕ_U and ϕ_V cause the following diagram to commute:

$$\begin{array}{ccc} \|\mathcal{D}/U\| & \xrightarrow{\phi_U} & \mathcal{W}(U) \\ \downarrow \|\mathcal{D}/i\| & & \downarrow \mathcal{W}(i) \\ \|\mathcal{D}/V\| & \xrightarrow{\phi_V} & \mathcal{W}(V) \end{array}$$

Consider the case when $U = \{1\} \amalg \emptyset$ and $V = \{1\} \amalg \{1\}$. Recall that the **2**-cube $\mathcal{W}(\{1\} \amalg -)$ is constant, so in particular for the inclusion $i : \{1\} \amalg \emptyset \hookrightarrow \{1\} \amalg \{1\}$, the map $\mathcal{W}(i)$ is the identity. Then the commutative square above shows that $\phi_U = \mathcal{W}(i)^{-1} \circ \phi_V \circ \|\mathcal{D}/i\|$. That is, given a $\phi_{\{1\} \amalg \{1\}}$, there is a unique $\phi_{\{1\} \amalg \emptyset}$ that corresponds to it. This means that the restriction map from $\text{holim}_{\mathcal{P}_0(S)}$ to $\text{holim}_{\mathcal{K}}$ is an isomorphism since the only map that is in the former that is not in the latter is $\phi_{\{1\} \amalg \emptyset}$. \square

Corollary 7.9. *For $n \geq 2$, the $(n+1)^{\text{st}}$ cross effect $cr_{n+1}F(X_1, \dots, X_{n+1})$ is equivalent to the iterated cross effect $cr_2(cr_n F(X_1, \dots, X_{n-1}, -))(X_n, X_{n+1})$.*

Proof. As noted prior to Lemma 7.7, the $(n+2)$ -cube \mathcal{W} that computes the iterated cross effect can be written as a 1-cube of $(n+1)$ -cubes:

$$\partial^T \mathcal{W} \rightarrow \partial_{\{1\} \amalg \emptyset} \mathcal{W}.$$

The cube $\partial^T \mathcal{W}$ is exactly the cube used to define the $(n+1)^{\text{st}}$ cross-effect of F , so

$$\text{hofib } \partial^T \mathcal{W} = cr_{n+1}F(X_1, \dots, X_{n+1}),$$

and the cube $\partial_{\{1\} \amalg \emptyset} \mathcal{W}$ is a cube of constant 2-cubes, so

$$\text{hofib } \partial_{\{1\} \amalg \emptyset} \mathcal{W} \simeq *.$$

Computing $\text{hofib } \mathcal{W} = cr_2(cr_n F(-, X_3, \dots, X_{n+1}))(X_1, X_2)$ by taking the homotopy fiber of these homotopy fibers gives us a natural map

$$cr_2(cr_n F(-, X_3, \dots, X_{n+1}))(X_1, X_2) \xrightarrow{\simeq} cr_{n+1}F(X_1, \dots, X_{n+1}).$$

\square

Corollary 7.10. *If the $(n+1)$ -cube defining $cr_{n+1}F(X_1, \dots, X_{n+1})$ is Cartesian, then taking the second cross effect with respect to any single X_i of the functor $cr_n F(X_1, \dots, X_n)$ results in a Cartesian 2-cube.*

Proof. Apply Lemma 3.18 to the result of the preceding Lemma 7.7 to reduce from an $(n+2)$ -cube to a 2-cube by taking fibers to compute the space cr_n from the n -cube defining it. \square

Another important feature of cross effects is that the cubes from which they are built have section maps to every structure map in the cube. This means that they are much nicer than arbitrary cubes; in particular, the homotopy group π_k of the total fiber can be computed from π_k of the vertices of the cube.

Hypothesis 7.11 (Compatible sections to structure maps.). We say that a T -cube \mathcal{X} has *compatible sections to all structure maps* if for each inclusion of subsets $i_{U,V} : U \hookrightarrow V$, there exists a section map $s_{V,U} : \mathcal{X}(V) \rightarrow \mathcal{X}(U)$, and furthermore these section maps compose so that $s_{W,V} \circ s_{V,U} = s_{W,U}$.

Lemma 7.12. *The cubes $\mathcal{X}_T^{\{X_i\}}$ used to construct the cross effects satisfy the compatible sections hypothesis (7.11).*

Proof. Explicitly, given $U \subset V$ and the induced projection $\bigvee_{u \notin U} X_u \rightarrow \bigvee_{v \notin V} X_v$, has a section map that is the identity on each X_v for $v \notin V$ (by hypothesis, $U \subset V$, so if v is not in V , then v is also not in U). It is easy to see that these are all compatible in the sense of 7.11. \square

Lemma 7.13. *The group (or set) $\pi_k \perp F(X)$ is isomorphic to the iterated fiber of the cube of groups (or sets) $\text{fib } \pi_k F\mathcal{X}$. Extending our definition of \perp to functors to groups or sets, this can be restated as: $\pi_k \perp F(X) = \perp \pi_k F(X)$; that is, \perp commutes with π_k .*

Proof. Recall that $\perp F(X)$ is the homotopy fiber of a cube $F\mathcal{X}$ that has compatible sections to all structure maps. Also, recall that the total homotopy fiber of a cube can be computed by iterating the process of taking fibers in one direction at a time.

The existence of sections means that the long exact sequences of the fibrations in one direction involved actually break up into short exact sequences for each π_k . This means that π_k of the homotopy fiber of each structure map $\mathcal{X}(i_{U,V})$ is the fiber of the map $\pi_k \mathcal{X}(i_{U,V})$. The fact that the sections are compatible means that they pass to sections on the fibers, so this argument can be iterated until the total fiber is reached. \square

7.3 Cotriples

(The introduction to cotriples in this section follows Weibel [26, Chapter 8.6].)

In homological algebra, one frequently uses the technique of forming a free resolution of an R -module M . The canonical functorial way of doing this is to begin by applying the free functor F to set of elements of the module M , producing $R[M]$ whose elements are formal sums $\sum r_i m_i$, and then mapping that to M by sending the formal element $r_i m_i$ to the element $r_i \cdot m_i$ given by letting r_i act on m_i . Iterating this construction produces an acyclic (in dimension > 0) chain of free R -modules, and hence a free resolution of M . Another way of looking at this construction is as result of iteratively applying the functor

$\perp = FU$, the composition of the adjoint pair consisting of the free R -module functor and the forgetful functor from R -modules to sets. An axiomatization of this approach creates the objects called “cotriples”.

The intent of a cotriple is to create a simplicial object that functions as a resolution of X . The simplicial object $RX = ([n] \mapsto \perp^{n+1} X)$ is equipped with a natural map $RX \rightarrow X$ (derived from $\perp \rightarrow \text{Id}$) that associates the resolution RX to the object X . (In general, RX is acyclic in positive degrees and $\pi_0 RX = X$.) In particular, if $X = \perp Y$ (e.g., X is already a free module), then this complex is homotopic to the constant simplicial object $\perp Y$, so iterating the construction of these resolutions is idempotent up to homotopy.

Precisely speaking, a cotriple is a functor \perp equipped with natural transformations $\delta : \perp \rightarrow \perp^2$ and $\epsilon : \perp \rightarrow \text{Id}$ such that the following diagrams commute:

$$\begin{array}{ccc} \perp & \xrightarrow{\delta} & \perp^2 \\ \downarrow \delta & & \downarrow \perp \delta \\ \perp^2 & \xrightarrow{\delta \perp} & \perp^3 \end{array} \quad \begin{array}{ccc} & \perp & \\ \swarrow = & \downarrow \delta & \searrow = \\ \perp & \perp^2 & \perp \\ \epsilon \perp \swarrow & & \searrow \perp \epsilon \end{array}$$

The notation of Section 7.2 is not a coincidence; the cross effect \perp_n is in fact a cotriple, with augmentation map ϵ induced by the fold map $\bigvee^n X \rightarrow X$, and the diagonal map δ induced by the diagonal inclusion of \mathbf{n} into $\mathbf{n} \times \mathbf{n}$. The proof is somewhat technical, so we illustrate the idea in Section 7.4 and prove it in Section 7.5.

7.4 Illustration: The Cross-Effects Form A Cotriple

This section contains an illustration of the idea of a proof that the functor \perp is a cotriple. The purpose of this section is to provide a plausible motivation for the somewhat technical proof contained in Section 7.5. From this illustration, the reader can see that There are very few ingredients needed to prove that \perp is a cotriple; this section gives the reader an idea what they are and how they could be assembled to form a proof. We only consider $\perp = \perp_2$ in this section.

Recall that a cotriple requires two commuting diagrams:

$$\begin{array}{ccc} \perp \perp & \xrightarrow{\perp \epsilon} & \perp \\ \downarrow \epsilon \perp & & \downarrow \epsilon \\ \perp & \xrightarrow{\epsilon} & 1 \end{array} \quad (7.14)$$

and

$$\begin{array}{c}
 \perp \\
 \swarrow \quad \downarrow \delta \quad \searrow = \\
 \perp \xleftarrow[\perp \epsilon]{=} \perp \xrightarrow[\epsilon \perp]{} \perp
 \end{array} \tag{7.15}$$

7.4.1 The Notation

This section uses some nonstandard, but very visually intuitive, notation. To understand the functors \perp and $\perp \perp$, we will consider them as the total fibers of 2- and 4-dimensional cubes, respectively. (Remember that we are only working with $\perp = \perp_2$ to keep the argument understandable to the reader.)

The space $\perp F(X)$ is the total homotopy fiber of the cube

$$\begin{array}{ccc}
 F(X \vee X) & \longrightarrow & F(X) \\
 \downarrow & & \downarrow \\
 F(X) & \longrightarrow & F(0)
 \end{array}$$

The argument we make is essentially independent of F , so we will omit the application of F to our cubes. (There is one exception to the assertion that F does not matter: at some points we need to consider 0 instead of $F(0)$.) That leaves us with the cubes:

$$\begin{array}{ccc}
 X \vee X & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 X & \longrightarrow & 0
 \end{array}$$

We will write subscripts on the spaces X to distinguish them. This has the effect of making it clear what the maps are: they are the identity on X_i and the zero map between spaces without the same subscript.

$$\begin{array}{ccc}
 X_1 \vee X_2 & \longrightarrow & X_1 \\
 \downarrow & & \downarrow \\
 X_2 & \longrightarrow & 0
 \end{array}$$

In order to make it possible to write four dimensional cubes, we engage in one more reduction of structure; we also omit the arrows entirely, writing the cubes in the form of matrices:

$$\begin{pmatrix} (X_1 & X_2) & X_1 \\ & X_2 & 0 \end{pmatrix}$$

When we write four dimensional cubes for $\perp \perp$, we will doubly index the spaces X as $X_{i,j}$. Our convention for the meaning of the indices in $X_{i,j}$ is that the first index, i , corresponds to the first application of the functor \perp (that is,

the rightmost \perp). The second index, j , corresponds to the second application of \perp (the left one).

7.4.2 Explicit Models For \perp_2 And $\perp_2 \perp_2$

In order to write down the maps δ , $\epsilon \perp$, and $\perp \epsilon$ explicitly, we will use explicit models for the cross effects. Recall that using Goodwillie's model for the total fiber, given a cube of cubes, taking homotopy fibers twice commutes up to natural homeomorphism. We will use this to blow up models for $\perp F(X)$.

As above, recall that in our notation, $\perp F$ is the homotopy fiber of F applied to the cube:

$$\begin{pmatrix} \begin{pmatrix} X_1 & X_2 \end{pmatrix} & X_1 \\ X_2 & 0 \end{pmatrix}$$

Note that the total fiber of the cube above is homeomorphic to the total fiber of the following 4-cube (2-cube of 2-cubes):

$$\begin{pmatrix} \begin{pmatrix} X_1 & X_2 \end{pmatrix} & X_1 \\ X_2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We can enlarge this to the following cube, which we will refer to as computing $\widetilde{\perp}_2$, by expanding some of the sub-cubes but maintaining the property that the total fiber of all of the cubes except that in the upper left is contractible.

$$\begin{pmatrix} \begin{pmatrix} X_1 & X_2 \end{pmatrix} & X_1 \\ X_2 & 0 \end{pmatrix} \quad \begin{pmatrix} X_1 & X_1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} X_2 & 0 \\ X_2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

There is a map $\widetilde{\perp} F(X) \rightarrow \perp F(X)$ induced by sending the vertices of all cubes except that in the upper right to zero. (Strictly speaking, apply F first, then map to 0. This causes the homotopy fibers of those cubes to have exactly one point, so the total fiber is homeomorphic to $\perp F(X)$.)

Given a T -cube \mathcal{X} and a function $f : S \rightarrow T$, there is an induced functor $\mathcal{P}(f) : \mathcal{P}(S) \rightarrow \mathcal{P}(T)$, and then this induces a map

$$\mathrm{hofib}_{\mathcal{P}(T)} \mathcal{X} \rightarrow \mathrm{hofib}_{\mathcal{P}(S)} \mathcal{P}(f)^* \mathcal{X}.$$

This can be used to produce a map from the homotopy fiber of the two-dimensional cube for $\perp F(X)$ to the homotopy fiber of the four-dimensional cube for $\widetilde{\perp} F(X)$; see Section 7.5.2 for details.

The functor $\perp \perp$ is computed by applying \perp twice; this naturally corresponds to the total fiber of the following 4-cube (2-cube of 2-cubes):

$$\begin{array}{ccc} \left(\begin{array}{cc} \begin{pmatrix} X_{11} & X_{21} \\ X_{12} & X_{22} \end{pmatrix} & \begin{pmatrix} X_{11} & X_{21} \end{pmatrix} \\ \begin{pmatrix} X_{12} & X_{22} \end{pmatrix} & 0 \end{array} \right) & \left(\begin{array}{cc} \begin{pmatrix} X_{11} \\ X_{12} \end{pmatrix} & \begin{pmatrix} X_{11} \end{pmatrix} \\ \begin{pmatrix} X_{12} \end{pmatrix} & 0 \end{array} \right) & (7.16) \\ \left(\begin{array}{cc} \begin{pmatrix} X_{21} & X_{22} \end{pmatrix} & \begin{pmatrix} X_{21} \end{pmatrix} \\ \begin{pmatrix} X_{22} \end{pmatrix} & 0 \end{array} \right) & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

The diagonal map $\delta : \perp \rightarrow \perp \perp$ is induced by the composition of the map $\perp \rightarrow \widetilde{\perp}$ with the map from $\widetilde{\perp}$ to $\perp \perp$ is induced by sending X_1 to X_{11} and X_2 to X_{22} .

There are two maps $\perp \perp \rightarrow \perp$. Recall that our convention for the meaning of the indices in $X_{i,j}$ is that the first index, i , corresponds to the first application of the functor \perp (that is, the rightmost \perp), and the second index, j , corresponds to the second application of \perp (here, the left one).

With this convention, the map $\perp \epsilon : \perp \perp \rightarrow \perp$ is induced by sending $X_{m,n}$ to X_m . To remind the reader of which spaces map to which, we will write the image as $X_{m,*}$ before identifying $X_{m,*}$ with X_m in the cube defining the single application of \perp . Formulating this in terms of cubical diagrams, $\perp \epsilon$ is induced by the map of $\perp \perp$ (7.16) to the following 4-cube, followed by taking the total fiber (which is now easy to see is homeomorphic to \perp).

$$\begin{array}{ccc} \left(\begin{array}{cc} \begin{pmatrix} X_{1,*} & X_{2,*} \end{pmatrix} & 0 \\ 0 & 0 \end{array} \right) & \begin{pmatrix} X_{1,*} & 0 \\ 0 & 0 \end{pmatrix} \\ \left(\begin{array}{cc} X_{2,*} & 0 \\ 0 & 0 \end{array} \right) & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

Although we have not illustrated this fact in this section, the map $\perp \rightarrow \widetilde{\perp}$ is a section to the map induced by the zero map on the spaces X_i not in the upper left sub-cube, so the composition $(\perp \epsilon)\delta = 1$, and hence the left hand triangle in (7.15) commutes.

The map $\epsilon \perp$ is similar. It is induced by mapping the 4-cube for $\perp \perp$ to the following:

$$\begin{pmatrix} \begin{pmatrix} X_{*,1} \\ X_{*,2} \end{pmatrix} & X_{*,1} \\ X_{*,2} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

As with the case of $\perp \epsilon$, we have the composition $(\perp \epsilon)\delta = 1$, so the right triangle in (7.15) commutes. Finally, we verify that the square in Equation (7.14) commutes. The identification of the image of $\perp \epsilon$ with \perp comes from identifying $X_{1,*}$ with X_1 and $X_{2,*}$ with X_2 . The identification of the image of $\epsilon \perp$ with \perp comes from identifying $X_{*,1}$ with X_1 and $X_{*,2}$ with X_2 . This means, for instance, that the space $X_{1,2}$ is identified with $X_{1,*}$ under the map $\perp \epsilon$, but identified with $X_{*,2}$ under $\epsilon \perp$, so these two maps are not the same as maps from $\perp \perp$ to \perp . The map ϵ is induced by mapping

$$\begin{pmatrix} \begin{pmatrix} X_1 & X_2 \end{pmatrix} & X_1 \\ X_2 & 0 \end{pmatrix}$$

to

$$\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}$$

Under this map, all $X_{m,n}$ in the upper left corner are identified, equalizing the images of $\epsilon \perp$ and $\perp \epsilon$, so the diagram in Equation (7.14) commutes.

7.5 Proof: The Cross-Effects Form A Cotriple

In this section we produce a formal proof that \perp is a cotriple. We require only a good model for the homotopy fiber of a cube, such as the one given in Definition 7.3, that is functorial in the indexing category.

We introduce the machinery of “free cubes” in order to have a good method of being precise about certain maps. Actually, the “free cubes” that we will use are like free modules over a ring. The “ring” U determines the order of the cross effect \perp_U being used. A $(k \times U)$ -cube (meaning the indexing category is the disjoint union of k copies of U) is the analogy of a rank k free module over the “ring” U . The rank k determines the number of iterations \perp_U^k .

7.5.1 Free Cubes

We will begin by defining a “free cube” with a given “generating function”. This requires that we build up a bit of notation.

Notation 7.17. Given a set U and a subset A of U , let A^c denote the complement of A in U .

We define a “diagonal” to encode the information needed to construct a cube of coproducts and inclusion and projection maps of the type used to define the cross effect.

Definition 7.18 (Diagonal). For any sets S and U , define the “diagonal” $\Delta(S, U)$ to be the subsets of $\mathcal{P}(S \times U)$ that are complements of a singleton in each component. That is, given a function $f : S \rightarrow U$, define the set $B_f \subset S \times U$ by

$$B_f = \bigcup_{s \in S} (s, f(s)^c),$$

and then define the diagonal $\Delta(S, U)$ to be

$$\Delta(S, U) = \bigcup_{f: S \rightarrow U} \{B_f\}.$$

Remark 7.19. When S is the empty set, there is one function $f : \emptyset \rightarrow U$, resulting in the empty set as the union over $s \in \emptyset = S$ being the only member of $\Delta(\emptyset, U)$. Also note that $\Delta(S, U) \cong \text{Hom}(S, U) \cong U^S$ via the correspondence $B_f \leftrightarrow f$.

Example 7.20. Let \mathcal{X} denote the T -cube $\mathcal{X}_T^{\{X_u\}}$ from Section 7.1 that forms a basis for defining the cross effect. Recall that $\mathcal{X}(U) = \bigvee_{u \notin U} X_u$. The “digaonal” sets $V \in \Delta(\mathbf{1}, T)$ are those sets for which $\mathcal{X}(V)$ consists only of a single space X_v for some $v \in T$.

Lemma 7.21. *The diagonal $\Delta(S, U)$ is contravariantly functorial in S and covariantly functorial in U .*

Proof. This is clear from the natural isomorphism $\Delta(S, U) \cong \text{Hom}(S, U)$, but we will give explicit maps that we will use later.

The diagonal $\Delta(S, U)$ is contravariantly functorial in S . Given a map $g : S \rightarrow T$, we can define a map $\Delta(g, 1) : \Delta(T, U) \rightarrow \Delta(S, U)$ sending $B \in \Delta(T, U)$ to $(g \times 1)^{-1}(B)$.

$$\Delta(g, 1)(B) = (g \times 1)^{-1}(B)$$

The set $(g \times 1)^{-1}(B)$ is still the complement of a singleton in each component, because for each $j \in S$, we have $\{j\} \times U \cong \{g(j)\} \times U$.

The diagonal $\Delta(S, U)$ is covariantly functorial in U . Given a function $h : U \rightarrow V$, define $\Delta(1, h)$ by “pushing the missed singletons along h ”:

$$\Delta(1, h)(B_f) = B_{h \circ f}$$

□

Definition 7.22 (Free cube). Given sets U and S and a functor g from the discrete category $\Delta(S, U)$ to a pointed category with coproducts (for example pointed spaces or cubes of pointed spaces), we define $\text{Free}(S, U, g)$ to be the $(S \times U)$ -cube \mathcal{X} with vertices

$$\mathcal{X}(A) = \bigvee_{\{B \in \Delta(S, U) : A \subset B\}} g(B)$$

Morphisms in \mathcal{X} are induced by the maps $g(B) \rightarrow g(B')$ that are the identity if $B = B'$ and the zero map otherwise.

Remark 7.23. It is easy to define a map out of a free cube, or between free cubes. Every point in a free cube is in the image of a section map of one of the spaces on the diagonal (that is, a space $\mathcal{X}(B)$ with $B \in \Delta(S, U)$), so it suffices to give a map from each space on the diagonal. From this, it is clear that $\text{Free}(S, U, g)$ is a covariant functor with respect to natural transformations of g .

Definition 7.24 (Alternative free cube). An alternative formulation of Definition 7.22 follows. This form of the definition of a free cube is useful because it more closely resembles the definition of the cross effect. For $A \subset S \times U$ and for $s \in S$, define the projection of A on the s factor, $p_s(A)$, to be the intersection of A with $\{s\} \times U$, considered as a subset of U . Then $A \subset B_f$ if and only if $f(s) \notin p_s(A)$ for all $s \in S$, so we can define $\text{Free}(S, U, g)$ to be the cube \mathcal{X} with vertices

$$\mathcal{X}(A) = \bigvee_{\{B_f \in \Delta(S, U) : \forall s \in S, f(s) \notin p_s(A)\}} g(B_f)$$

Example 7.25. The n^{th} cross effect, $cr_n F(X_1, \dots, X_n)$, is the total fiber of F applied to a free cube \mathcal{X} . Let $\mathbf{n} = \{1, \dots, n\}$, and define $\mathcal{X} = \text{Free}(\mathbf{1}, \mathbf{n}, g)$, with $g(\{i\}^c) = X_i$. When $n = 2$, we can write this down very explicitly. First,

$\Delta(\mathbf{1}, \mathbf{2}) = \{\{1\}, \{2\}\}$. Then we can compute:

$$\begin{aligned}\mathcal{X}(\emptyset) &= \bigvee_{B \in \Delta(\mathbf{1}, \mathbf{n})} g(B) \\ &= g(\{1\}) \vee g(\{2\}) \\ &= g(\{2\}^c) \vee g(\{1\}^c) \\ &= X_2 \vee X_1\end{aligned}$$

Similarly, $\mathcal{X}(\{1\}) = X_2$, $\mathcal{X}(\{2\}) = X_1$, and $\mathcal{X}(\{1, 2\}) = 0$, so the cube is exactly the cross-effect cube we wanted to see.

In general, given a functor \mathcal{X} defined on a category \mathcal{D} and another functor $F : \mathcal{C} \rightarrow \mathcal{D}$, we can define the pullback functor $F^*\mathcal{X}$ precomposing with F . When dealing with cubical diagrams, a function $f : S \rightarrow T$ induces a functor $\mathcal{P}(f) : \mathcal{P}(S) \rightarrow \mathcal{P}(T)$; in this case, the pullback operation is $\mathcal{P}(f)^*$.

Similarly, given sets n and m , and a function $f : n \rightarrow m$ between them, there is an induced map of sets $f \times 1 : n \times U \rightarrow m \times U$ that can then be used to define a functor $\mathcal{P}(f \times 1) : \mathcal{P}(n \times U) \rightarrow \mathcal{P}(m \times U)$. In this case, we will denote the pullback by $\mathcal{P}(f \times 1)^*$.

We now establish that “free cubes” are closed under the pullback operation.

Lemma 7.26. *Let m and n be sets, let the $(m \times U)$ -cube $\mathcal{X} = \text{Free}(m, U, g)$ be a free cube, and let $f : n \rightarrow m$ be a function. The $(n \times U)$ -cube $\mathcal{P}(f \times 1)^*\mathcal{X}$ is isomorphic to a free cube $\mathcal{Y} = \text{Free}(n, U, h)$ with*

$$h(B) = \bigvee_{B' \in \Delta(f, 1)^{-1}(B)} g(B').$$

Proof. We consider the question one vertex at a time. Fix a subset $A \subset n \times U$.

$$\mathcal{Y}(A) = \bigvee_{\{B \in \Delta(n, U) : A \subset B\}} h(B)$$

Expanding the definition of h gives:

$$\bigvee_{\{B \in \Delta(n, U) : A \subset B\}} \bigvee_{\{B' \in \Delta(f, 1)^{-1}(B)\}} g(B')$$

Interchanging the order of quantifiers and combining them turns this into:

$$\bigvee_{\{B' \in \Delta(m, U) : A \subset \Delta(f, 1)(B')\}} g(B')$$

We now show that the indexing set

$$\{B' \in \Delta(m, U) : A \subset \Delta(f, 1)(B')\}$$

is the same as

$$\{B' \in \Delta(m, U) : (f \times 1)A \subset B'\}.$$

The latter is the indexing set for $\mathcal{P}(f \times 1)^*\mathcal{X}$, so this will establish that $\mathcal{Y} \cong \mathcal{P}(f \times 1)^*\mathcal{X}$. This is elementary set theory. There are two directions to show containment. Recall that $\Delta(f, 1)(B') = (f \times 1)^{-1}(B')$. If $A \subset \Delta(f, 1)B'$, then this means:

$$\begin{aligned} A &\subset (f \times 1)^{-1}B' \\ (f \times 1)A &\subset (f \times 1)(f \times 1)^{-1}B' \subset B', \end{aligned}$$

where $(f \times 1)(f \times 1)^{-1}B'$ may be smaller than B' if some components are not in the image (i.e., if f is not surjective). This establishes containment in one direction. On the other hand, if $(f \times 1)A \subset B'$, then applying $(f \times 1)^{-1}$ gives

$$A \subset (f \times 1)^{-1}(f \times 1)A \subset (f \times 1)^{-1}B' = \Delta(f, 1)(B'),$$

which establishes containment in the other direction. \square

7.5.2 Homotopy Fibers

In this section we briefly recall from Bousfield-Kan [9, XI, §9, p. 316] the map on homotopy fibers induced by a functor on diagram categories. The purpose of this section is to prove the following proposition (which follows immediately from this work of Bousfield and Kan, as indicated below):

Proposition 7.27. *Let $f : S \rightarrow T$ be a map of sets, and let \mathcal{X} be a T -cube of pointed spaces. Then f induces a natural map*

$$\mathrm{hofib}_{\mathcal{P}(T)} \mathcal{X} \rightarrow \mathrm{hofib}_{\mathcal{P}(S)} \mathcal{P}(f)^* \mathcal{X},$$

and if f is surjective, then this map is a homotopy equivalence.

Given sets S and T and a function $f : S \rightarrow T$, there is an induced functor on the power set categories: $\mathcal{P}(f) : \mathcal{P}(S) \rightarrow \mathcal{P}(T)$. Since the inverse image of the empty set is the empty set, this functor restricts to a functor $\mathcal{P}_0(f)$ from $\mathcal{P}_0(S)$ to $\mathcal{P}_0(T)$. Let \mathcal{X} be a T -cube; that is, a functor whose domain category is $\mathcal{P}(T)$. One definition of the homotopy fiber of \mathcal{X} is strict fiber in the following fiber sequence:

$$\mathrm{hofib}_{\mathcal{P}(T)} \mathcal{X} \rightarrow \mathrm{holim}_{\mathcal{P}(T)} \mathcal{X} \rightarrow \mathrm{holim}_{\mathcal{P}_0(T)} \mathcal{X}$$

This shows that in order to produce a map $\mathrm{hofib}_{\mathcal{P}(T)} \mathcal{X} \rightarrow \mathrm{hofib}_{\mathcal{P}(S)} \mathcal{P}(f)^* \mathcal{X}$, it suffices

to produce the two right vertical maps in the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{hofib}_{\mathcal{P}(T)} \mathcal{X} & \longrightarrow & \mathrm{holim}_{\mathcal{P}(T)} \mathcal{X} & \longrightarrow & \mathrm{holim}_{\mathcal{P}_0(T)} \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{hofib}_{\mathcal{P}(S)} \mathcal{P}(f)^* \mathcal{X} & \longrightarrow & \mathrm{holim}_{\mathcal{P}(S)} \mathcal{P}(f)^* \mathcal{X} & \longrightarrow & \mathrm{holim}_{\mathcal{P}_0(S)} \mathcal{P}(f)^* \mathcal{X}
\end{array}$$

This is the classical situation from Bousfield and Kan. It is just as easy to describe in the general setting, so we do so. Given categories \mathcal{C} and \mathcal{D} and a functor \mathcal{X} on \mathcal{D} and a functor $F : \mathcal{C} \rightarrow \mathcal{D}$, we want to produce a map

$$\mathrm{holim}_{\mathcal{D}} \mathcal{X} \rightarrow \mathrm{holim}_{\mathcal{C}} F^* \mathcal{X}$$

That is, produce a function

$$\mathrm{Hom}_{\mathcal{D}}(\|\mathcal{D}/-\|, \mathcal{X}(-)) \rightarrow \mathrm{Hom}_{\mathcal{C}}(\|\mathcal{C}/-\|, \mathcal{X} \circ F(-))$$

Essentially, this follows from the contravariance of Hom and the fact that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ induces a simplicial map $\|\mathcal{C}/c\| \rightarrow \|\mathcal{D}/F(c)\|$ for all objects c in \mathcal{C} .

Specifically, the elements on the left of the diagram above are coherent collections of maps ϕ_d sending n -simplices corresponding to $d \leftarrow d_1 \leftarrow \dots \leftarrow d_n$ to $\mathcal{X}(d)$. Given one of these, define a function ϕ_c on $\|\mathcal{C}/c\|$ by sending the n -simplex $c \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_n} c_n$ to the same place as the n -simplex in $\mathcal{D}/F(c)$ corresponding to its image under F :

$$\phi_{F(c)}(F(c) \xleftarrow{F(\alpha_1)} \dots \xleftarrow{F(\alpha_n)} F(c_n)).$$

Note that the target of this function is $\mathcal{X}F(c) = F^* \mathcal{X}(c)$, just as required. Coherence of the collection $\{\phi_c\}$ follows from that of the collection $\{\phi_d\}$.

If the map $f : S \rightarrow T$ is surjective, then $\mathcal{P}_0(f)$ is a “left cofinal” functor from $\mathcal{P}_0(S)$ to $\mathcal{P}_0(T)$, and hence [9, §XI.9, Theorem 9.2] the induced map between homotopy fibers is a homotopy equivalence. To verify that in this case $\mathcal{P}_0(f)$ is left cofinal, we need to check that for each $V \subset T$, a certain category $\mathcal{P}_0(f)/V$ is contractible. The objects in this category are elements (U, μ) , where $U \subset S$ and $\mu : f(U) \rightarrow V$ is a morphism in $\mathcal{P}_0(T)$; that is, an inclusion of $f(U)$ into V . In the power set category, the maps are inclusions of subsets, so there can be at most one map between objects; hence the category $\mathcal{P}_0(f)/V$ has one element for each subset of $f^{-1}(V)$, and maps correspond to inclusions. All subsets of $f^{-1}(V)$ include in $f^{-1}(V)$, so $\mathcal{P}_0(f)/V$ is contractible whenever $f^{-1}(V)$ is nonempty. This shows that if f is surjective, then $\mathcal{P}_0(f)$ is left cofinal.

If $f^{-1}(V)$ is empty for some V (that is, when f is not surjective), then the category $\mathcal{P}_0(f)/V$ is empty, and the empty space is not contractible (that is, \emptyset is not equivalent to $*$), so $\mathcal{P}_0(f)$ is not left cofinal in that case.

This completes the proof of Proposition 7.27. \square

7.5.3 \perp As Composition With Free Cube Functor

The purpose of developing the machinery of “free cubes” was to enable us to show that \perp is a cotriple. In this section, we identify the functor \perp as a composition of functors with the free cube functor. Throughout this section, we fix a set U and a functor F and a space X , and consider only $\perp_U F(X)$.

Given sets U and S , let c_X be the function on $\Delta(S, U)$ that has a constant value X . Let $C(S)$ be the contravariant functor of sets S given by

$$C(S) = \text{hofib } F \circ \text{Free}(S, U, c_X).$$

Functoriality. We need to establish that this is a functor. Let $\mathcal{Y} = \text{Free}(T, U, c_X)$. Given a function $f : S \rightarrow T$, we can construct the $(S \times U)$ -cube $\mathcal{P}(f \times 1)^* \mathcal{Y}$. From Proposition 7.27, there is a natural map

$$\text{hofib}_{\mathcal{P}(T)} F \mathcal{Y} \rightarrow \text{hofib}_{\mathcal{P}(S)} \mathcal{P}(f \times 1)^* \mathcal{Y},$$

so it remains to construct a map

$$\text{hofib}_{\mathcal{P}(S)} F \circ \mathcal{P}(f \times 1)^* \mathcal{Y} \rightarrow \text{hofib}_{\mathcal{P}(S)} F \circ \text{Free}(S, U, c_X).$$

Recall from Lemma 7.26 that $\mathcal{P}(f \times 1)^* \mathcal{Y}$ is a free cube with generating function

$$h(B) = \bigvee_{\Delta(f, 1)^{-1}(B)} X.$$

To map this to the constant function c_X , we use the fold map from $h(B) = \bigvee X$ to $c_X(B) = X$. If $h(B)$ is the empty coproduct, 0, then the fold map is the inclusion of 0 in X . As noted in Remark 7.23, since h is the “generating function” for $\mathcal{P}(f \times 1)^* \mathcal{Y}$, specifying maps $h(B) \rightarrow c_X(B)$ suffices to determine a map of cubes

$$\mathcal{P}(f \times 1)^* \mathcal{Y} \rightarrow \text{Free}(S, U, c_X).$$

The homotopy fiber functor is covariant with respect to maps of cubes, so applying F and hofib give the required map

$$\text{hofib}_{\mathcal{P}(S)} F \circ \mathcal{P}(f \times 1)^* \mathcal{Y} \rightarrow \text{hofib}_{\mathcal{P}(S)} F \circ \text{Free}(S, U, c_X).$$

Identification with \perp . Let \mathbf{k} denote the set $\{1, \dots, k\}$. To identify $C(\mathbf{k})$ with $\perp^k F(X)$, we recall from Example 7.6 that $\perp^k F(X)$ is

$$\text{hofib}_{\{V_1, \dots, V_k\} \subset U} F \left(\bigvee_{i=1}^{i=k} \bigvee_{v_i \notin V_i} X \right)$$

The cube involved is a $(\mathbf{k} \times U)$ -cube \mathcal{X} . Given $V \subset (\mathbf{k} \times U)$, for each $i \in \mathbf{k}$ define V_i to be the projection of $V \cap \{i\} \times U$ to U . In the notation of Definition 7.24, $V_i = p_i(V)$. The cube then has vertices

$$\mathcal{X}(V) = \bigvee_{i=1}^{i=k} \bigvee_{v_i \notin V_i} X.$$

From the alternative construction of a “free cube” in Definition 7.24, we see that this is exactly a free cube $\text{Free}(\mathbf{k}, U, c_X)$, with c_X a constant function whose value is X on all $B_f \in \Delta(\mathbf{k}, U)$. This shows that $C(\mathbf{k}) \cong \perp_U^k F(X)$.

Claim: The map $\epsilon : \perp \rightarrow 1$ is induced by applying C to the inclusion $i : \emptyset \rightarrow 1$. Let $\mathcal{X} = \text{Free}(1, U, c_X)$. According to Section 7.5.2, the homotopy fiber of the cube $F\mathcal{X}$ is a coherent collection of maps

$$\text{hofib}_{\mathcal{P}(U)} F\mathcal{X} = \{\phi_c : ||\mathcal{C}/c|| \rightarrow F\mathcal{X}(c)\}_{c \in \text{Obj } \mathcal{C}}$$

The map from the homotopy fiber of $F\mathcal{X}$ over $\mathcal{P}(U)$ to the homotopy fiber of the restriction of F over $\mathcal{P}(\emptyset)$ sends each map $\phi = \{\phi_c\}$ to the map ϕ_\emptyset induced by the image of the point $\emptyset \rightarrow \emptyset$ in \mathcal{C}/\emptyset ; this is just a single point $\phi(*)$ in \mathcal{X} . That is, i^* induces the identity on $\mathcal{X}(\emptyset) = X \vee X$.

Definition: The map $\delta : \perp \rightarrow \perp \perp$ is induced by the applying C to the fold map $\{1, 2\} \rightarrow \{1\}$. The map δ has not been specified any other way elsewhere in this work.

7.5.4 \perp Is A Cotriple

We are now in a position to use this machinery to show that \perp forms a cotriple. Fix a set U and a functor F , and consider only iterates of \perp_U applied to F , and evaluated at a fixed space X . That is, we only work with $\perp_U^k F(X)$. This is sufficient to show that \perp_U is a cotriple.

A cotriple \perp requires that the following diagram commute:

$$\begin{array}{ccc} \perp \perp & \xrightarrow{\perp \epsilon} & \perp \\ \downarrow \epsilon \perp & & \downarrow \epsilon \\ \perp & \xrightarrow{\epsilon} & 1 \end{array}$$

Applying the functor C to the diagram

$$\begin{array}{ccc} \{1, 2\} & \xleftarrow{i_1} & \{1\} \\ i_2 \uparrow & & \uparrow \\ \{2\} & \xleftarrow{\quad} & \emptyset \end{array}$$

yields a commuting diagram

$$\begin{array}{ccc} C(\{1, 2\}) & \longrightarrow & C(\{1\}) \\ \downarrow & & \downarrow \\ C(\{2\}) & \longrightarrow & C(\emptyset) \end{array}$$

In view of our identification $C(\mathbf{k}) \cong \perp_U^k F(X)$, this is the diagram above.

The other commuting diagram required for a cotriple is the following:

$$\begin{array}{ccccc} & & \perp & & \\ & \swarrow & \downarrow \delta & \searrow & \\ \perp & \xleftarrow{\quad \perp \epsilon \quad} & \perp & \xrightarrow{\quad \epsilon \perp \quad} & \perp \end{array}$$

This results from applying C to the diagram of sets:

$$\begin{array}{ccccc} & & \{1\} & & \\ & \nearrow & \uparrow & \nwarrow & \\ \{1\} & \xrightarrow{i_1} & \{1, 2\} & \xleftarrow{i_2} & \{2\} \end{array}$$

To summarize, we have shown:

Theorem 7.28. *The functor \perp of Definition 7.4 is a cotriple on the category of homotopy functors from pointed spaces to pointed spaces. \square*

Chapter 8

Properties Of P_n^d And \perp_n

Recall that the n -additive approximation functor, P_n^d is given by

$$P_n^d F(X) = P_n(LF_X)(S^0),$$

where L is the left Kan extension over all finite sets and $F_X(Y) = F(X \wedge Y)$.

In this chapter, we prove basic properties about the functors P_n^d and \perp_n . In order to be able to work effectively with P_n^d , we need to restrict the functors under consideration to those that give us some control of the behavior on π_0 . To do this, we introduce two hypotheses that a functor may satisfy in order for our results to be applicable.

Hypothesis 8.1 (Connected Values). F has connected values (on coproducts of X) if the functor F has the property that for spaces X under consideration, $F(\bigvee X)$ is connected for all finite coproducts of X .

Hypothesis 8.2 (Group Values). In the following definition, let \mathcal{T} denote the category of pointed spaces, and let \mathcal{C} denote the full subcategory of \mathcal{T} generated by all finite coproducts of S^0 . Let \mathcal{G} denote the category of topological groups, and let $U : \mathcal{G} \rightarrow \mathcal{T}$ be the forgetful functor.

F is group-valued (on coproducts of X) if there exists a functor F' so that the following diagram commutes:

$$\begin{array}{ccc} & & \mathcal{G} \\ & \nearrow F' & \downarrow U \\ \mathcal{C} & \xrightarrow{F(X \wedge -)} & \mathcal{T} \end{array}$$

In this case, we will conflate $F(X \wedge -)$ with its lift to groups.

Functors that satisfy Hypothesis 8.1 (connected values) or Hypothesis 8.2 (group values) on coproducts of X are the subject of the first two sections in this chapter. Section 8.1 gives conditions under which the functor P_n^d preserves fibrations of functors. Section 8.2 establishes a fundamentally important lemma

for working with the approximations P_n^d : they preserve connectivity of (good) natural transformations of functors.

The last section of this chapter (§8.3) contains a proof of a technical result showing that under good circumstances if the total fiber of a cube is contractible, then it is Cartesian. This technical lemma makes it possible to deduce useful information from the functor \perp_n .

8.1 P_n^d Preserves (Group Or Connected) Fibrations

In this section, we give conditions under which P_n^d preserves fibrations. This lets us understand the effects of P_n^d when we understand the decomposition of a functor as a part of a fibration over some other functor.

Proposition 8.3. *Given a space X and functors A , B , and C , suppose $A(Y) \rightarrow B(Y) \rightarrow C(Y)$ is a fibration sequence for all finite coproducts $Y = \bigvee X$ of X . If, on finite coproducts of X , either:*

1. *C takes connected values (Hypothesis 8.1); or*
2. *B and C take group values (Hypothesis 8.2), and the map $B \rightarrow C$ is a surjective homomorphism of groups,*

then

$$LA_X(Z) \rightarrow LB_X(Z) \rightarrow LC_X(Z) \quad (8.4)$$

is a fibration sequence for all spaces Z . Furthermore, the sequence is surjective on π_0 .

Proof. Equation (8.4) is equivalent to

$$\|A(X \wedge Z.)\| \rightarrow \|B(X \wedge Z.)\| \rightarrow \|C(X \wedge Z.)\|, \quad (8.5)$$

and since $Z.$ is discrete, $C(X \wedge Z.) = C(\bigvee X)$, for some coproduct of X . In the case of Hypothesis 8.1, the base space is always connected, so Waldhausen's Lemma (Lemma 2.10) tells us that the fiber of the realization is the realization of the fibers, so (8.4) is a fibration. In the case of Hypothesis 8.2, the total space and the base space are simplicial groups, and hence satisfy the π_* -Kan condition (2.11). Furthermore, since we have assumed that the map $B \rightarrow C$ is surjective, the second map in (8.5) is surjective levelwise. In particular, a surjective map of simplicial groups is a fibration, so this map is a fibration on π_0 . That allows us to apply Theorem 2.12 (Bousfield-Friedlander) to conclude that (8.4) is a fibration. Surjectivity on π_0 follows from noting that the realization of a levelwise 0-connected map is 0-connected. \square

Corollary 8.6. *Under the conditions of Proposition 8.3,*

$$P_n^d A(X) \rightarrow P_n^d B(X) \rightarrow P_n^d C(X) \quad (8.7)$$

is a fibration sequence.

Proof. By Proposition 8.3, applying $L(-)_X$ to the sequence $A \rightarrow B \rightarrow C$ yields a fibration sequence of functors (*i.e.*, a fibration when evaluated at any space). Applying P_n preserves this fibration, as does evaluation at S^0 . That is the definition of P_n^d . \square

Remark 8.8. Notice that this argument does not show that the resulting fibration sequence is surjective on π_0 . Even P_1 need not preserve connectivity of maps (or spaces). For example, the functor $F(X) = \Sigma\Omega X$ always produces 0-connected spaces, but $P_1 F(X) \simeq QX$ need only be (-1) -connected. (F does not increase the connectivity of 0-connected spaces, just (-1) -connected spaces.)

8.2 P_n^d Preserves Connectivity Of Natural Transformations

In this section, we establish a property of fundamental importance when working with P_n^d : the n -additive approximation preserves the connectivity of natural transformations that satisfy some basic good properties.

Theorem 8.9. *Let F and G be functors that have connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X . Let $\eta : F \rightarrow G$ be a natural transformation of functors, and suppose that η lifts to a homomorphism of groups on π_0 in the case of Hypothesis 8.2. If η is k -connected on coproducts of X , then $P_n^d \eta$ is k -connected.*

First let us recall some more of Goodwillie's calculus machinery. For our purposes the action of Σ_n is not particularly important, but we describe it here for completeness. This exposition is based on Goodwillie's lectures in Aberdeen, Scotland, June 18–23, 2001.

Before we begin, the definition of a derivative will require an action of Σ_n on Ω^{n-1} . To write this, we will let V_n denote the standard representation of Σ_n on \mathbb{R}^n . Let \overline{V}_n denote the representation of Σ_n on \mathbb{R}^{n-1} created by splitting the one-dimensional trivial representation off of V_n . Let S^V be the one-point compactification of V . Let Ω^V denote $\text{Map}(S^V, -)$. For $k \in \mathbb{N}$, let $k \cdot V$ denote the product of V with itself k times.

Definition 8.10 (Derivative of F). The n^{th} derivative of F (at $*$), denoted $\partial^{(n)} F(*)$, is the following spectrum with Σ_n action, which we will denote \mathbf{Y} . The space Y_k in the spectrum is $\Omega^{k \cdot \overline{V}_n} cr_n F(S^k, \dots, S^k)$. The structure map

$Y_k \rightarrow \Omega Y_{k+1}$ arises from applying T_1 in each variable of cr_n (that is, suspending inside and looping outside). Given any symmetric functor of n variables $H(A, \dots, A)$, this procedure produces a Σ_n -equivariant map $H(A, \dots, A) \rightarrow \Omega^n H(S^1 \wedge A, \dots, S^1 \wedge A)$. Here Ω^n is Ω^{V_n} , so we can split off the trivial representation and write this as a map

$$H(A, \dots, A) \rightarrow \Omega \Omega^{\overline{V_n}} H(S^1 \wedge A, \dots, S^1 \wedge A).$$

Using $H = \Omega^{k \cdot \overline{V_n}} cr_n F$ and $A = S^k$ produces the desired map $Y_k \rightarrow \Omega Y_{k+1}$.

We will only really be interested in the nonequivariant homotopy type; for our application we only need to understand the spectrum \mathbf{Y} with $Y_k = \Omega^{k(n-1)} \perp_n F(S^k)$.

When F satisfies the limit axiom (5.1), we can express $D_n F(X)$ using the derivative:

$$D_n F(X) \simeq \Omega^\infty \left(\partial^{(n)} F(*) \wedge_{h\Sigma_n} X^{\wedge n} \right). \quad (8.11)$$

In order to prove Theorem 8.9, we first want to establish that $\partial^{(n)} LF_X(*)$ is a connective spectrum for all n . Note that LF_X always satisfies the limit axiom (Lemma 5.6).

Lemma 8.12. *If F is a reduced functor that has connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X , then for all n , the derivative $\partial^{(n)} LF_X(*)$ of LF_X is a connective spectrum.*

Proof. Lemma 8.17 essentially shows that L and \perp commute; we have deferred the proof to the end of this section so as not to get bogged down in technicalities. Using Lemma 8.17, we can compute $\perp_n LF_X(S^k) \simeq \perp_n \|F(X \wedge S^k)\|$ by computing $\|\perp_n F(X \wedge S^k)\|$. Now $\perp_n F(X \wedge S^k) = cr_n F(X \wedge S^k, \dots, X \wedge S^k)$, and $cr_n F$ is contractible if any one of its inputs is contractible, so by the Eilenberg-Zilber Theorem (2.8), $\|\perp_n F(X \wedge S^k)\|$ is $(nk - 1)$ -connected. Subtracting $k(n - 1)$ from $nk - 1$ shows that $\Omega^{k(n-1)} \perp_n (LF_X)(S^k)$ is $(k - 1)$ -connected.

To determine the connectivity of the spectrum $\mathbf{Y} = \partial^{(n)} LF_X(*)$, we compute the colimit:

$$\begin{aligned} \pi_m \mathbf{Y} &= \operatorname{colim}_k \pi_{m+k} Y_k \\ &= \operatorname{colim}_k \pi_{m+k} \Omega^{k(n-1)} \perp_n (LF_X)(S^k). \end{aligned}$$

We have just shown that the space $\Omega^{k(n-1)} \perp_n (LF_X)(S^k)$ is $(k - 1)$ -connected, so $\pi_m \mathbf{Y} = 0$ for $m < 0$. That is, $\partial^{(n)} LF_X(*) = \mathbf{Y}$ is a connective spectrum. \square

Corollary 8.13. *If F is a reduced functor that has connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X , then for all $n \geq 1$, the map $P_{n+1}^d F(X) \rightarrow P_n^d F(X)$ is surjective in π_0 .*

Proof. Theorem 3.14 (Goodwillie's delooping of D_n) shows that the fibration

$$D_{n+1}LF_X \rightarrow P_{n+1}LF_X \rightarrow P_nLF_X$$

deloops to a fibration

$$P_{n+1}LF_X \rightarrow P_nLF_X \rightarrow \Omega^{-1}D_{n+1}LF_X.$$

The delooping of $D_{n+1}LF_X$ consists of smashing with the suspension of $\partial^{(n+1)}LF_X(*)$ and taking homotopy orbits. By Lemma 8.12, the spectrum $\partial^{(n+1)}LF_X(*)$ is connective, so its suspension is 0-connected; hence $\pi_0\Omega^{-1}D_{n+1}LF_X = 0$, so evaluation at S^0 shows that the map $P_{n+1}^dF(X) \rightarrow P_n^dF(X)$ is surjective on π_0 . \square

Lemma 8.14. *Let F and G be reduced functors, and suppose that either F and G have connected values (Hypothesis 8.1) on coproducts of X , or F and G have group values (Hypothesis 8.2) on coproducts of X . If $\eta : F \rightarrow G$ is a natural transformation that is w -connected on coproducts of X , then for $n \geq 1$, the natural transformation $D_n^d\eta$ is w -connected.*

Proof. For any functor H , we know

$$\begin{aligned} D_n^dH(X) &= D_n(LH_X)(S^0) \\ &= \Omega^\infty \left(\partial^{(n)}LH_X(*) \wedge_{h\Sigma_n} (S^0)^{\wedge n} \right). \end{aligned}$$

Taking homotopy orbits and smashing with a fixed space preserves connectivity, so this is really a question about the connectivity of the map $\partial^{(n)}LF_X(*) \rightarrow \partial^{(n)}LG_X(*)$.

First, consider the case $n = 1$. Since $\eta : F \rightarrow G$ is w -connected on coproducts of X , the map $\|F_X(S^k)\| \rightarrow \|G_X(S^k)\|$ is $(k + w)$ -connected (by Eilenberg-Zilber, since both are contractible levelwise until dimension k). The derivative spectrum $\partial LF_X(*)$ has k^{th} space $\|F_X(S^k)\|$, and similarly for $\partial LG_X(*)$, so this shows that the map $\partial LF_X(*) \rightarrow \partial LG_X(*)$ is a w -connected map.

Similarly, for all $n \geq 1$, the map $\|\perp_n F_X(S^k)\| \rightarrow \|\perp_n G_X(S^k)\|$ is $(nk + w)$ -connected. The derivative spectrum $\partial^{(n)}LF_X(*)$ then has as its k^{th} space the space $\Omega^{k(n-1)}\|\perp_n F_X(S^k)\|$. On these spaces the map induced by η is $(k + w)$ -connected, exactly as required to produce a w -connected map $\partial^{(n)}LF_X(*) \rightarrow \partial^{(n)}LG_X(*)$. \square

Corollary 8.15. *Let F and G be reduced functors, and suppose that either F and G have connected values (Hypothesis 8.1) on coproducts of X , or F and G have group values (Hypothesis 8.2) on coproducts of X . If $\eta : F \rightarrow G$ is a natural transformation that is w -connected on coproducts of X , then for $n \geq 1$, the natural transformation $P_n^d\eta$ is w -connected.*

Proof. We induct up the Goodwillie tower for F and G . Our hypotheses on the values of F and G , along with Corollary 8.13 provide the control on π_0 needed to use the Five Lemma, making this an easy consequence of Corollary 8.14. \square

Proof of Theorem 8.9. First we will show that we may reduce to the case of reduced functors. Let $F_0(X) = F(0)$ and $G_0(X) = G(0)$ be constant functors, and consider the fibration

$$\begin{array}{ccc} \tilde{F} & \longrightarrow & \tilde{G} \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \\ \downarrow & & \downarrow \\ F_0 & \longrightarrow & G_0 \end{array}$$

By Corollary 8.6, applying P_n^d preserves these fibrations. The map $F \rightarrow F_0$ and $G \rightarrow G_0$ have sections induced by $0 \rightarrow X$, so applying P_n^d also preserves surjectivity on π_0 .

The approximation P_n^d applied to a constant functor is just the constant functor again, so the map on the bases is k -connected. Corollary 8.15 shows that the induced map $P_n^d \tilde{F} \rightarrow P_n^d \tilde{G}$ is k -connected. The Five Lemma applies in this situation (connected or group values, surjective on π_0), allowing us to conclude that the map on the total spaces is k -connected. \square

Theorem 8.9 gives us the following slight but essential improvement of Corollary 8.6:

Corollary 8.16. *Given a space X and functors A , B , and C , suppose $A(Y) \rightarrow B(Y) \rightarrow C(Y)$ is a fibration sequence for all finite coproducts $Y = \bigvee X$ of X . If, on finite coproducts of X , either:*

1. *C takes connected values (Hypothesis 8.1); or*
2. *B and C take group values (Hypothesis 8.2), and the map $B \rightarrow C$ is a surjective homomorphism of groups,*

then

$$P_n^d A(X) \rightarrow P_n^d B(X) \rightarrow P_n^d C(X)$$

is a fibration sequence that is surjective on π_0 . \square

We have deferred the proof that \perp commutes with realizations until this point. It is straightforward.

Lemma 8.17. *If F has connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X , then*

$$cr_n(LF_X)(Y^1, \dots, Y^n) \simeq L^n(cr_n F)_{(X, \dots, X)}(Y^1, \dots, Y^n).$$

For each j , let Y^j be a simplicial set whose realization is Y^j . Then

$$L^n(cr_n F)_{(X, \dots, X)}(Y^1, \dots, Y^n) \simeq ||(cr_n F)_{(X, \dots, X)}(Y^1, \dots, Y^n)||,$$

where the realization is taken in each of the n dimensions involved, and the realization can be the “strict” realization with no adverse effect on the homotopy type of the result.

The statement above can be abbreviated to $\perp_n(LF_X)(Y) \simeq L(\perp_n F)_X(Y)$.

Proof. The space $cr_n(LF_X)(Y^1, \dots, Y^n)$ is the homotopy fiber of a cube involving $LF_X(\bigvee_{u \in U} Y^u)$ for $U \subset \{1, \dots, n\}$. We will write this part of our argument assuming $U = \{1, \dots, n\}$ for simplicity. By Proposition 4.9, this is the realization of the simplicial space

$$F_X(\text{diag}(Y^1 \vee \dots \vee Y^n)) = \text{diag } F_X(Y^1 \vee \dots \vee Y^n).$$

Lemma 4.10 shows that this is a good space, so we can use the strict realization. Then the Eilenberg-Zilber theorem (2.8) shows that the diagonal has the same homotopy type as the (multidimensional) realization of the n -dimensional simplicial space $F_X(Y^1 \vee \dots \vee Y^n)$.

Now if we show that we can take the fibers of the maps before applying the realization functor, we will be able to interpret the fibers of each dimension (k_1, \dots, k_n) as $(cr_n F)_{(X, \dots, X)}(Y_{k_1}^1, \dots, Y_{k_n}^n)$, which is the left Kan extension in each variable, as desired for the lemma.

If F satisfies Hypothesis 8.1, then we can compute the fibers in the \perp -cube levelwise using Waldhausen’s Lemma (Lemma 2.10).

If F satisfies Hypothesis 8.2, then we will use Theorem 2.12 (Bousfield-Friedlander) to produce the same result. In this case, F_X also satisfies Hypothesis 8.2 on coproducts of S^0 (i.e., on all finite sets). This shows that for any simplicial set Y , we may regard the simplicial space $LF_X(Y) \simeq ||F_X(Y)||$ as a simplicial group. Hence each corner of the \perp -cube satisfies the π_* -Kan condition. Furthermore, all of the maps in the \perp -cube have compatible sections, so at each stage of taking iterated fibers all of the structure maps have sections. This gives us surjective maps of simplicial groups, so the induced maps on π_0 are fibrations. These two conditions are enough to apply Theorem 2.12 to compute the fibers levelwise. \square

8.3 Fiber Contractible Implies Cartesian (Group Or Connected)

We will prove the critical fact that in the cases we consider, the cross effect vanishing is equivalent to the cross effect cubes being Cartesian.

We generally want to use the fact that the cross effect is contractible to

conclude that the initial space in the cross-effect cube is equivalent to the (homotopy) inverse limit of the rest of the spaces. Unfortunately, this is not always true; the problem is that the homotopy fiber does not detect failure to be surjective on π_0 . Fortunately, with some mild hypotheses we are able to ensure that we stay within the realm where this issue is avoided for one reason or another.

This section lays the groundwork that shows that in good situations, the cross-effect cubes are well behaved and total fiber contractible implies Cartesian. Section 8.3.1 considers the case of a functor to connected spaces. Section 8.3.2 considers the case when the functor is group-valued.

If the spectral sequence developed by Bousfield and Kan in [9] were shown to converge to π_0 under these conditions, it could be used to make the arguments in this section shorter. Their work would give $\pi_k \operatorname{holim}_{\mathcal{P}_0(S)} \mathcal{X} = \lim_{\mathcal{P}_0(S)} \pi_k \mathcal{X}$ with higher \lim^i terms vanishing because of the structure of the cube used to compute \perp .

8.3.1 Connected Values

Even if all of the spaces in the cross-effect cube are connected, we need to know that the homotopy inverse limit of the punctured cube (“the rest of the spaces” $= \mathcal{P}_0(S)$) is still connected in order to be able to conclude that cross effect zero implies the cube is Cartesian.

To show that the homotopy inverse limit of the punctured cube is connected, we proceed as follows: first, we show that a pullback of a diagram of connected spaces with section maps is connected; then we decompose the whole homotopy inverse limit into (iterated) pullbacks of diagrams of this form and diagrams with initial objects.

Given a cube of spaces with compatible sections, the first thing we want to do is replace all of the maps in the cube by fibrations so that the homotopy inverse limit is equivalent to the strict inverse limit.

Lemma 8.18. *If \mathcal{X} is a cube of spaces with compatible sections to all structure maps (7.11), then \mathcal{X} is equivalent to a cube \mathcal{X}' of spaces in which all structure maps are fibrations, and all of these maps still have compatible sections.*

Proof. From [18, Remark 1.14, p. 305], every cube of spaces is equivalent to a fibration cube by replacing $\mathcal{X}(U)$ by $\mathcal{Y}(U) = \operatorname{holim}_{U \subset V} \mathcal{X}(V) = \operatorname{holim}(\partial_U \mathcal{X})$. The maps in the cube are then induced by the inclusion of indexing categories. The section maps in \mathcal{X} can then be used to give section maps in \mathcal{Y} .

We will now briefly sketch an example showing how to construct the section maps. The reader may wish to review the definition of the precise construction of homotopy inverse limit that we are using before proceeding.

Given a map $f : c \rightarrow d$ in $\mathcal{C} = \mathcal{P}(S)$, and a map $g : \|d \setminus \mathcal{C} / d\| \rightarrow \mathcal{X}(d)$ (representing a point in $\operatorname{holim}(\partial_{\{d\}} \mathcal{X})$), produce an element in $\operatorname{holim}(\partial_{\{c\}} \mathcal{X})$ as follows: collapse $\|c \setminus \mathcal{C} / d\|$ to the image of $\|d \setminus \mathcal{C} / d\|$, then use g to map

to $\mathcal{X}(d)$. On $\|c \setminus \mathcal{C} / c\|$, first map a point to its image in $\|c \setminus \mathcal{C} / d\|$, then to $\mathcal{X}(d)$, and then via the section map to $\mathcal{X}(c)$. The general case is obviously very similar, just harder to write down explicitly. \square

Now we establish that if we have section maps, the pullback of connected spaces is connected.

Lemma 8.19. *Let \mathcal{X} be an $\mathbf{2}$ -cube with compatible sections to all structure maps (7.11). If each space $\mathcal{X}(U)$ is connected, then so is $\text{holim}_{\mathcal{P}_0(\mathbf{2})} \mathcal{X}$.*

Proof. By Lemma 8.18, we may assume that \mathcal{X} is a fibration cube. Then the homotopy inverse limit is equivalent to the strict inverse limit, so we need only show that if

$$\mathcal{X} = (X \xrightarrow{p_X} Z \xleftarrow{p_Y} Y)$$

is a diagram with sections s_X and s_Y to the maps p_X and p_Y , and all three spaces are connected, then so is their inverse limit.

A map f of the 0-sphere to the inverse limit is equivalent to compatible maps f_X , f_Y , and f_Z of S^0 to all three spaces. We first show that $f = (f_X, f_Z, f_Y)$ is homotopic to the map $f' = (s_X f_Z, f_Z, s_Y f_Z)$, and then use a homotopy in Z to show f' is null homotopic.

Since Z is connected, the fibers of the map $p_X : X \rightarrow Z$ over every point are equivalent. Due to the section s_X , the fibers over every point in Z are connected (existence of the section map implies surjectivity on π_* , so connectivity of the fiber cannot drop). Let $1 \in S^0$ denote the non-basepoint element of the 0-sphere. The points $f_X(1)$ and $s_z f_Z(1)$ are in the same fiber over the point $f_Z(1)$, and this fiber is connected, so there is a homotopy $H_X : f_X \simeq s_z f_Z$ that stays entirely within the fiber (so $p_X H_X$ is the constant map f_Z).

By the symmetry of X and Y , this shows that the map $f = (f_X, f_Z, f_Y)$ is homotopic to the map f' with components $(s_X f_Z, f_Z, s_Y f_Z)$. Now let $H : D^1 \rightarrow Z$ be a homotopy $f_Z \simeq *$. Then the homotopy $(s_X H, H, s_Y H)$ is a null homotopy of f' .

This shows that $\pi_0 \text{holim}_{\mathcal{P}_0(\mathbf{2})} \mathcal{X} = *$, as required for the lemma. \square

We can now use Proposition 3.17 to decompose the inverse limit of the punctured cube into pullbacks and inverse limits with initial objects.

Lemma 8.20. *Let \mathcal{X} be an S -cube with compatible sections to all structure maps (7.11). If each space $\mathcal{X}(U)$ is connected, then so is $\text{holim}_{\mathcal{P}_0(S)} \mathcal{X}$.*

Proof. As indicated, our approach is to use Proposition 3.17 to write $\text{holim}_{\mathcal{P}_0(S)} \mathcal{X}$ as pullbacks and spaces that are vertices of \mathcal{X} . As in Lemma 8.18, when this is done in a natural way, all of the maps between the inverse limits will have sections, so we will be able to apply the result for the case of a pullback with sections, Lemma 8.19, repeatedly to conclude that the whole inverse limit is connected.

It is well known how to produce an inverse limit of the type used here by iterating pullbacks, so here we will just sketch the method used. To produce a homotopy pullback from Proposition 3.17, we need to produce a decomposition of $\mathcal{P}_0(S)$. Let $\mathcal{A}(U)$ be the full subcategory of $\mathcal{P}_0(S)$ generated by $\{V \subset S \mid U \subset V\}$. Begin the decomposition by considering $\mathcal{A}(U)$ for U a maximal element, and the union of $\mathcal{A}(V)$ over all of the other maximal elements $V \neq U$. The sets are finite and the maximum height of the maximal elements decreases in the intersection, so proceeding inductively we end up with the base case of a pullback in from Lemma 8.19. \square

Example 8.21. As an example of the decomposition of the homotopy inverse limit in Lemma 8.20, consider the S -cube \mathcal{X} with $S = \{1, 2\}$. We have two maximal elements in $\mathcal{P}_0(S)$: $\{1\}$ and $\{2\}$, so our decomposition is $A(1) = (\{1\} \rightarrow \{1, 2\})$ and $A(2) = (\{2\} \rightarrow \{1, 2\})$. Proposition 3.17 gives $\text{holim}_{\mathcal{P}_0(S)} \mathcal{X}$ equal to the homotopy pullback of

$$\text{holim}_{A(1)} \mathcal{X} \rightarrow \text{holim}_{A(1) \cap A(2)} \mathcal{X} \leftarrow \text{holim}_{A(2)} \mathcal{X}.$$

Now $\text{holim}_{A(1)} \mathcal{X} \simeq \mathcal{X}(1)$, since $\{1\}$ is initial in $A(1)$, and similarly $\text{holim}_{A(2)} \mathcal{X} \simeq \mathcal{X}(2)$, and $\text{holim}_{A(1) \cap A(2)} \mathcal{X} \simeq \mathcal{X}(\{1, 2\})$.

Example 8.22. A more complicated example is the case of a 3-cube, where we begin with $A(1)$ and $A(2) \cup A(3)$. Then their intersection is $(A(1) \cap A(2)) \cup (A(1) \cap A(3))$. This category is

$$\{1, 2\} \rightarrow \{1, 2, 3\} \leftarrow \{1, 3\},$$

so the inverse limit over it is a pullback. Then one proceeds to decompose $A(2) \cup A(3)$ into $A(2)$ and $A(3)$ in a manner similar to the 2-cube from the previous example.

Knowing that the homotopy limit is connected is the key piece of information to conclude that the \perp -cube is Cartesian when the total fiber is contractible.

Lemma 8.23. *Let F be a functor satisfying Hypothesis 8.1 (connected values) on coproducts of X . If $\perp_n F(X) \simeq 0$, then the cube defining $\perp_n F(X)$ is Cartesian.*

Proof. Let \mathcal{C} be the cube defining $\perp_n F(X)$. Lemma 8.20 shows that $\text{holim}_{U \in \mathcal{P}_0(\mathbf{n})} \mathcal{C}(U)$ is connected. The cross effect is the homotopy fiber in the quasifibration

$$\perp_n F(X) \rightarrow \mathcal{C}(\emptyset) \rightarrow \text{holim}_{U \in \mathcal{P}_0(\mathbf{n})} \mathcal{C}(U).$$

The cube \mathcal{C} is Cartesian if the right map is an equivalence. A map is an equivalence if it has a contractible homotopy fiber and is surjective on π_0 . The fiber is $\perp_n F(X) \simeq 0$, and the base has $\pi_0 = *$, so the map is an equivalence. \square

8.3.2 Group Values

In this section, we establish that for a functor to groups, $\perp F(X) \simeq 0$ implies that the cube defining $\perp F(X)$ is Cartesian.

Lemma 8.24. *Let \mathcal{X} be an n -cube ($n \geq 1$) of discrete groups. If \mathcal{X} has compatible sections to all structure maps (7.11), then the map*

$$\mathcal{X}(\emptyset) \rightarrow \lim_{U \in \mathcal{P}_0(\mathbf{n})} \mathcal{X}(U)$$

is surjective.

Proof. We need to show that the map above is surjective. This is equivalent to showing that there exists an $x_\emptyset \in \mathcal{X}(\emptyset)$ mapping to each coherent system of elements $x_U \in \mathcal{X}(U)$, with $U \neq \emptyset$.

\mathcal{X} is a cube of groups, and hence all of the structure maps are group homomorphisms. This allows us to subtract an arbitrary $w \in \mathcal{X}(\emptyset)$ from x_\emptyset , and subtract the images $\text{Im}_U(w)$ of w in $\mathcal{X}(U)$ from each x_U , to show the question is equivalent to the existence of an $x_\emptyset - w \in \mathcal{X}(\emptyset)$ mapping to each coherent system of elements $x_U - \text{Im}_U(w)$.

Given a coherent system of elements x_U in an n -cube, let w be the image of $x_{\{n\}}$ in $\mathcal{X}(\emptyset)$ using the section map $\mathcal{X}(\{n\}) \rightarrow \mathcal{X}(\emptyset)$. Define $z_U = x_U - \text{Im}_U(w)$, noting that when $\{n\} \subset V$, we have $\text{Im}_V(w) = x_W$, so $z_W = 0$. By the preceding paragraph, the surjectivity that we are trying to establish is equivalent to the existence of a z_\emptyset mapping to each coherent collection z_U .

If $n = 1$, then $\lim_{U \in \mathcal{P}_0(\mathbf{1})} \mathcal{X}(U) = \mathcal{X}(\{1\})$, so the section map $\mathcal{X}(\{1\}) \rightarrow \mathcal{X}(\emptyset)$ produces a z_\emptyset mapping to $z_{\{1\}}$, as desired.

If $n > 1$, then we proceed by induction, assuming the lemma is true for smaller n . Taking the fiber of \mathcal{X} in the direction of $\{n\}$, we have an $(n-1)$ -cube

$$\mathcal{Y}(U) := \text{fib}(\mathcal{X}(U) \rightarrow \mathcal{X}(U \cup \{n\})).$$

The cube \mathcal{Y} satisfies the hypothesis of the lemma because taking fibers preserves compatible sections. Notice that for $\{n\} \not\subset U$, the element z_U passes to the fiber, since it maps to $z_{U \cup \{n\}} = 0$.

Now \mathcal{Y} is an $(n-1)$ -cube, so by induction, the map from $\mathcal{Y}(\emptyset)$ to $\lim \mathcal{Y}(U)$ is surjective. That is, there exists a $y \in \mathcal{Y}(\emptyset)$ with $\text{Im}_U(y) = z_U$. Mapping y to $z \in \mathcal{X}(\emptyset)$ gives an element z with $\text{Im}_U(z) = z_U$ for $U \subset \{1, \dots, n-1\}$. As above, if $\{n\} \subset U$, then $z_U = 0$, so $\text{Im}_U(z) = z_U$ in this case as well. Therefore, we have produced an element z mapping to each coherent collection of elements z_U , as desired. \square

Corollary 8.25. *Let \mathcal{X} be an n -cube ($n \geq 1$) of discrete groups with compatible sections to all structure maps (7.11). If the total fiber of \mathcal{X} is zero, then \mathcal{X} is Cartesian.*

Proof. Recall that a T -cube \mathcal{X} is Cartesian when $\mathcal{X}(\emptyset) \rightarrow \operatorname{holim}_{U \neq \emptyset} \mathcal{X}(U)$ is an equivalence. The hypotheses of the existence of section maps means that all of the structure maps are fibrations. This means that the homotopy inverse limit is equivalent to the strict inverse limit, so we need only show that $\mathcal{X}(\emptyset) \xrightarrow{\cong} \lim_{U \neq \emptyset} \mathcal{X}(U)$. Corollary 8.25 shows that the map is surjective. The total fiber of the cube is equivalent to the fiber of this map, so if the fiber is zero then we have a surjective map of groups with zero kernel; that is, an isomorphism. \square

The combination of Lemma 8.24 and Lemma 8.23 allow us to conclude that for all group-valued functors $\perp F \simeq 0$ means the \perp -cube is Cartesian.

Lemma 8.26. *Let F be a functor satisfying Hypothesis 8.2 (group values) on coproducts of X . If $\perp_n F(X) \simeq 0$, then the cube defining $\perp_n F(X)$ is Cartesian.*

Proof. Decompose F as the fibration

$$\widehat{F} \rightarrow F \rightarrow \pi_0 F,$$

where \widehat{F} is the connected component of the basepoint of F . Let \mathcal{X} be the $(n+1)$ -cube used in Definition 7.1 to define $\perp_n F(X)$.

In general, $X \rightarrow \pi_0 X$ has a section given by choosing a point in each component. Using the canonical sections to the structure maps in the cube \mathcal{X} , we may make compatible choices for these maps $\pi_0 \mathcal{X}(U) \rightarrow \mathcal{X}(U)$ so that they assemble to a map of cubes $\pi_0 \mathcal{X} \rightarrow \mathcal{X}$. We will use this in two places later in the proof.

We claim that if $\perp_n F(X)$ is contractible, then $\perp_n \widehat{F}(X)$ and $\perp_n \pi_0 F(X)$ are both contractible as well. The section map of cubes $\pi_0 F \mathcal{X} \rightarrow F \mathcal{X}$ produces a section $\perp \pi_0 F(X) \rightarrow \perp F(X)$. This shows that in the long exact sequence on homotopy associated to the fibration

$$\perp \widehat{F}(X) \rightarrow \perp F(X) \rightarrow \perp \pi_0 F(X),$$

there is a surjective map $\pi_k \perp F(X) \rightarrow \pi_k \perp \pi_0 F(X)$, for all k . Since $\pi_k \perp F(X) = 0$, this shows $\perp \pi_0 F(X)$ is contractible. Similarly, $\perp \widehat{F}(X)$ is now the fiber in a quasifibration whose total space and base space are contractible, so it is contractible as well.

The functor \widehat{F} satisfies Hypothesis 8.1 on coproducts of X , so by Lemma 8.23, the cube $\widehat{F} \mathcal{X}$ defining $\perp_n \widehat{F}(X)$ is Cartesian. The functor $\pi_0 F$ is a functor to discrete groups, so Corollary 8.25 shows that the cube $\pi_0 F \mathcal{X}$ defining $\perp_n \pi_0 F(X)$ is Cartesian. This shows that the left and right vertical arrows in the following diagram are equivalences:

$$\begin{array}{ccccc} \widehat{F} \mathcal{X}(\emptyset) & \longrightarrow & F \mathcal{X}(\emptyset) & \longrightarrow & \pi_0 F \mathcal{X}(\emptyset) \\ \downarrow \simeq & & \downarrow & & \downarrow \simeq \\ \operatorname{holim}_{\mathcal{P}_0} \widehat{F} \mathcal{X} & \longrightarrow & \operatorname{holim}_{\mathcal{P}_0} F \mathcal{X} & \longrightarrow & \operatorname{holim}_{\mathcal{P}_0} \pi_0 F \mathcal{X} \end{array}$$

The top row is a fibration by the construction of \widehat{F} . The bottom row is also a fibration, since homotopy inverse limits commute up to natural equivalence.

Consider the long exact sequence on homotopy groups. From the construction of \widehat{F} , we know that $\pi_0 \widehat{F} = 0$, so $\pi_0 \widehat{F}\mathcal{X}(\emptyset) = 0$, and hence $\pi_0(\operatorname{holim}_{\mathcal{P}_0} \widehat{F}\mathcal{X})$ is also 0. Furthermore, $F \rightarrow \pi_0 F$ is an isomorphism on π_0 .

Using the above data in the long exact sequences from the horizontal fibrations gives us the following diagram with exact rows:

$$\begin{array}{ccccc}
0 & \longrightarrow & \pi_0 F\mathcal{X}(\emptyset) & \xrightarrow{\cong} & \pi_0(\pi_0 F\mathcal{X}(\emptyset)) \\
\downarrow & & \downarrow & & \downarrow \cong \\
0 & \longrightarrow & \pi_0 \operatorname{holim}_{\mathcal{P}_0} F\mathcal{X} & \longrightarrow & \pi_0 \operatorname{holim}_{\mathcal{P}_0} \pi_0 F\mathcal{X}
\end{array}$$

Either by inverting the isomorphisms and using the center vertical map or by applying $\operatorname{holim}_{\mathcal{P}_0}$ to the map of cubes $\pi_0 F\mathcal{X} \rightarrow F\mathcal{X}$, we have a section to the bottom right map in the diagram, so it is surjective. By exactness, we have a surjective map of groups (by our hypothesis on F) that has kernel zero, so it is an isomorphism. \square

Chapter 9

The Main Theorem

In this chapter, we establish the fundamental theorem that makes the use of a cotriple workable for functors from spaces to spaces.

9.1 Main Theorem: Statement And Outline

The main theorem of this paper is the following.

Theorem 9.1. *Let F be a homotopy functor from pointed spaces to pointed spaces. If F has either connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X , then the following is a fibration sequence up to homotopy:*

$$||\perp_{n+1}^{*+1} F(X)|| \rightarrow F(X) \rightarrow P_n^d F(X). \quad (9.2)$$

To establish this theorem, we use induction on n , beginning with the case $n = 1$. We further break down the induction into the cases where $\perp_{n+1} F(X) \simeq 0$ and $\perp_{n+1} F(X) \not\simeq 0$.

In Section 9.2, we treat the case when $\perp_{n+1} F(X) \simeq 0$. In this case, we show directly that the fiber of the fibration sequence we obtain from induction,

$$||\perp_n^{*+1} F(X)|| \rightarrow F(X) \rightarrow P_{n-1}^d F(X),$$

is a homogeneous degree n functor. This implies that $F(X) \simeq P_n^d F(X)$ in this case.

In Section 9.3, we treat the case when $\perp_{n+1} F(X) \not\simeq 0$. In this case, we consider the auxiliary diagram:

$$\begin{array}{ccccc} A_F(X) & \longrightarrow & ||\perp_{n+1}^{*+1} F(X)|| & \xrightarrow{\epsilon} & F(X) \\ \downarrow & & \downarrow & & \downarrow \\ P_n^d A_F(X) & \longrightarrow & P_n^d (||\perp_{n+1}^{*+1} F(X)||) & \longrightarrow & P_n^d F(X) \end{array}$$

where A_F is defined as the homotopy fiber of the map ϵ in the top row, and the bottom row is shown to be a quasifibration as well (Proposition 9.30 and 9.31).

We show that $\perp_{n+1} A_F(X) \simeq 0$ (Lemma 9.29), and hence the case $\perp_{n+1} F \simeq 0$ shows that there is an equivalence of the fibers, so the square on the right is Cartesian. Then it is not hard to establish that $P_n^d (||\perp_{n+1}^{*+1} F(X)||) \simeq 0$ (Lemma 9.26), so that (9.2) is actually a quasifibration.

9.2 Case: $\perp F \simeq 0$

In this section, the goal is to establish that when the $(n+1)$ -st cross effect of F vanishes, F is equivalent to its n -additive approximation, $P_n^d F$.

Proposition 9.3. *If F has either connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X , and $\perp_{n+1} F(X) \simeq 0$, then $F(X) \simeq P_n^d F(X)$.*

Remark 9.4. Under our hypotheses, there is no difference between $\perp_{n+1} F(X)$ being contractible and the cube defining $\perp_{n+1} F(X)$ being Cartesian. (See §9.2.1.)

Outline of Proposition 9.3. We will approach this proposition using a ladder induction, depending on previous cases of Proposition 9.3 and Proposition 9.11. We begin with the classical case $n = 1$ (Corollary 9.7). It turns out that this case is essentially the work of Segal [23]: when $\perp_2 F(X) \simeq 0$, the fold map makes $F(X)$ into a (homotopy) monoid, and $P_1^d F(X)$ is naturally equivalent to loops on the bar construction of $F(X)$.

We then consider $n > 1$ and apply Proposition 9.11, considering the n^{th} cross effect, $\perp_n F(X)$, rather than the $(n+1)^{\text{st}}$ cross effect $\perp_{n+1} F(X)$. This gives us the fibration:

$$||\perp_n^{*+1} F(X)|| \rightarrow F(X) \rightarrow P_{n-1}^d F(X).$$

The problem is now to show that the fiber is a homogeneous degree n functor, $D_n^d F(X)$.

Since $\perp_{n+1} F(X)$ vanishes, we know that as a functor of n variables, $\perp_n F(X) = cr_n F(X, \dots, X)$ has zero second cross effect in each variable. Hence we may apply the $n = 1$ case to identify $\perp_n F(X)$ as the infinite loop space of a functor to connective spectra, $\perp_n \mathbf{F}(X)$ (Corollary 9.6). Then actually the entire simplicial space $||\perp_n^{*+1} F(X)||$ is an infinite loop space, $\Omega^\infty ||\perp_n^{*+1} \mathbf{F}(X)||$. From this point (in Lemma 9.9), we identify the simplicial spectrum as the homotopy orbits of $\perp_n^{*+1} \mathbf{F}$,

$$||\perp_n^{*+1} \mathbf{F}(X)|| \simeq \perp_n \mathbf{F}(X)_{h\Sigma_n},$$

which is a homogeneous functor, $D_n \mathbf{F}(X)$. It remains to show that $F(X) \simeq P_n^d F(X)$. Before evaluation at S^0 , our fibration can be written

$$D_n(LF_X) \rightarrow LF_X \rightarrow P_{n-1}(LF_X),$$

and in this form, the base and the fiber are n -excisive. Modulo the ever-present technical problem of π_0 , the property of n -excision is closed under extensions, so LF_X is n -excisive, so $LF_X \simeq P_n LF_X$. Evaluating at S^0 gives $F(X) \simeq P_n^d F(X)$. \square

9.2.1 $\perp F$ -cube Cartesian

If F has connected values (8.1) (or group values (8.2)), on coproducts of X , then Lemma 8.23 (respectively, Lemma 8.26) shows that $\perp_{n+1} F(X) \simeq 0$ implies that the cube defining $\perp_{n+1} F(X)$ is Cartesian, so henceforth we may assume that the \perp -cubes we are dealing with are Cartesian.

9.2.2 Additivization And The Bar Construction

If F is nice enough and $\perp_2 F(X) \simeq 0$, then actually $F(X)$ is a (homotopy) monoid (Γ -space), and $F(X)$ is equivalent to loops on the bar construction on $F(X)$.

Lemma 9.5. *Suppose F is a reduced functor that has either connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X . If $\perp_2 F(X) \simeq 0$, then one can form the bar construction $BF(X) = LF_X(S^1)$ on $F(X)$, and $BF(X)$ is connected, and $F(X) \simeq \Omega BF(X)$.*

Proof. Under these hypotheses, if $\perp_2 F(X) \simeq 0$, then the cube

$$\begin{array}{ccc} F(X \vee X) & \longrightarrow & F(X) \\ \downarrow & & \downarrow \\ F(X) & \longrightarrow & F(0) \simeq 0 \end{array}$$

is Cartesian, so $F(X \vee X) \simeq F(X) \times F(X)$. Furthermore, with this identification, the map $\epsilon : F(X \vee X) \rightarrow F(X)$ induces a the structure of a homotopy monoid on $F(X)$. This is the setting in which Segal's theory of Γ -spaces applies [23, Proposition 1.4], showing that $F(X) \simeq \Omega BF(X)$ and $BF(X) \simeq ||F(X \wedge S^1)||$ is connected. \square

Corollary 9.6. *If F has either connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X , and $\perp_{n+1} F(X) \simeq 0$, then, as a symmetric functor of n variables, $\perp_n F(X, \dots, X)$ is the infinite loop space of a symmetric functor to connective spectra $\perp_n \mathbf{F}(X, \dots, X)$:*

$$\perp_n F(X, \dots, X) \simeq \Omega^\infty (\perp_n \mathbf{F}(X, \dots, X)).$$

Proof. As usual, under these hypotheses, the cube defining $\perp_{n+1} F(X)$ is Cartesian (§9.2.1). By Corollary 7.10, the functor of n variables $\perp_n F(X, \dots, X)$ is also additive in each variable, and $\perp_n F(X, \dots, X)$ is always reduced in each

variable, so Lemma 9.5 gives $\perp_n F(X, \dots, X)$ as the first space of a connective Ω -spectrum. \square

Corollary 9.7. *If F is a reduced functor that has either connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X , and $\perp_2 F(X) \simeq 0$, then $F(X) \simeq P_1^d F(X)$.*

Proof. Lemma 9.5 shows that $F(X) \simeq \Omega BF(X)$. Continuing the argument from that lemma, consider one iteration of the functor $T_1^d F(X) \simeq \Omega \|F(S^1 \wedge X)\|$ used to build $P_1^d = \text{colim}_n (T_1^n)^d$. Actually, since $(T_1^n)^d$ is not necessarily $(T_1^d)^n$, we want to show $T_1 LF_X \simeq LF_X$ in order to show that $P_1^d F(X) \simeq F(X)$. Consider an arbitrary space $Y = \|Y\|$; we show $T_1 LF_X(Y) \simeq LF_X(Y)$.

$$\begin{aligned} T_1 LF_X(Y) &\simeq \Omega \|F(X \wedge (S^1 \wedge Y))\| \\ &\simeq \Omega \| \|F(X \wedge S^1 \wedge Y)\| \| \\ &\simeq \Omega \| \| [k] \mapsto F(X \wedge Y \wedge S_k^1) \| \| \\ &\simeq \Omega \| \| [k] \mapsto F(\bigvee_k X \wedge Y) \| \| \end{aligned}$$

And, since $X \wedge Y$ is a coproduct of copies of X , the cross effect $\perp_2 F(X \wedge Y) \simeq 0$, so this is

$$\begin{aligned} &\simeq \Omega \| \| [k] \mapsto \prod_k F(X \wedge Y) \| \| \\ &\simeq \Omega \| BF(X \wedge Y) \| \\ &\simeq \Omega B \|F(X \wedge Y)\| \\ &\simeq \Omega B LF_X(Y) \\ &\simeq LF_X(Y) \end{aligned}$$

So in fact, $T_1 LF_X(Y) \simeq LF_X(Y)$, and hence $P_1^d F(X) = \text{colim}(T_1^k)^d F(X) = \text{colim} T_1^k (LF_X)(S^0) \simeq LF_X(S^0) \simeq F(X)$, as desired. \square

9.2.3 Iterated Cross Effects Produce Homogeneous Functors

In this section, we write \perp instead of \perp_n , for convenience. We write Σ_n^+ for the space Σ_n with a disjoint basepoint added.

Let $\mathbf{H}(X_1, \dots, X_n)$ be a symmetric functor from spaces to spectra, and suppose \mathbf{H} is additive in each variable separately. In practice, such an \mathbf{H} will arise as $\perp \mathbf{F}$ from Corollary 9.6.

Using the additivity in each variable, we can identify $\perp_n \mathbf{H}$ with $\Sigma_n^+ \wedge \mathbf{H}$. Here we have $X_1 = \dots = X_n$, but label them differently to be able to see the

action of the symmetric group more clearly.

$$\begin{aligned}\perp \mathbf{H}(X_1, \dots, X_n) &\simeq \prod_{\alpha \in \Sigma_n} \mathbf{H}(X_{\alpha(1)}, \dots, X_{\alpha(n)}) \\ &\simeq \bigvee_{\alpha \in \Sigma_n} \mathbf{H}(X_{\alpha(1)}, \dots, X_{\alpha(n)}) \\ &\simeq \Sigma_n^+ \wedge \mathbf{H}(X_1, \dots, X_n).\end{aligned}$$

The second equivalence is the stable equivalence of finite coproducts and products, and the third equivalence is given by the map

$$\Sigma_n^+ \wedge \mathbf{H}(X_1, \dots, X_n) \rightarrow \bigvee_{\alpha \in \Sigma_n} \mathbf{H}(X_{\alpha(1)}, \dots, X_{\alpha(n)})$$

sending $\sigma \wedge x$ to x in the coproduct indexed by σ . To identify $x \in \mathbf{H}(X_1, \dots, X_n)$ with $x \in \mathbf{H}(X_{\alpha(1)}, \dots, X_{\alpha(n)})$, we use the map induced by $X_{\alpha(i)} \cong X_i$ in each variable.

The identification

$$\perp \mathbf{H}(X_1, \dots, X_n) \simeq \Sigma_n^+ \wedge \mathbf{H}(X_1, \dots, X_n) \quad (9.8)$$

can be made equivariant with respect to the action of Σ_n on both \mathbf{H} and $\perp(-)$ in the following way. The action induced by permuting the inputs of $\perp \mathbf{H}$ (*i.e.*, from the fact that $\perp(-)$ is a symmetric functor) is sent to multiplication on the Σ_n factor. The action on $\perp \mathbf{H}$ induced by the Σ_n action on \mathbf{H} is sent to the same action on \mathbf{H} on the other side.

Under this model, the map $\epsilon : \perp \mathbf{H} \rightarrow \mathbf{H}$ is given by $\sigma \wedge x \mapsto x$.

We are now in a position to understand $\perp^* \perp \mathbf{F}$. Applying (9.8) repeatedly at each level, we have

$$\perp^k \perp \mathbf{F}(X) \simeq \underbrace{\Sigma_n^+ \wedge \dots \wedge \Sigma_n^+}_{k \text{ factors}} \wedge \perp \mathbf{F}(X).$$

Recall that the face maps from dimension n to $n-1$ are given by $d_i = \perp^i \epsilon \perp^{n-i}$. In dimension k , the face map $d_k = \epsilon \perp^k$ just drops the first element:

$$d_k(g_k \wedge \dots \wedge g_1 \wedge y) = g_{k-1} \wedge \dots \wedge g_1 \wedge y.$$

To compute the others, note that for any f , the map $\perp(f)$ is equivariant with respect to the action of Σ_n on \perp (by permuting inputs), so in particular $\perp(\epsilon) : \perp(\perp F) \rightarrow \perp F$ is equivariant with respect to the action on Σ_n on the leftmost

\perp , so

$$\begin{aligned}\perp \epsilon(g \wedge y) &= \perp \epsilon(g * (1 \wedge y)) \\ &= g * \perp \epsilon(1 \wedge y) \\ &= g * y,\end{aligned}$$

where the last follows since the degeneracy $\delta : \perp F \rightarrow \perp^2 F$ given by $\delta(y) = 1 \wedge y$ is a section to the face map $\perp \epsilon$. This argument shows that all of the face maps d_j with $0 \leq j < k$ are given by multiplying g_{j+1} by the next coordinate to the right (either g_j if $j > 0$ or y if $j = 0$).

This is a standard model for $E\Sigma_n^+ \wedge_{\Sigma_n} \perp \mathbf{F}(X)$, so we have shown that the simplicial spectrum built by iterating the cross effects computes the homotopy orbits of $\perp \mathbf{F}(X)$. That is,

$$||\perp^* \perp \mathbf{F}(X)|| \simeq \perp \mathbf{F}(X) \wedge_{\Sigma_n} E\Sigma_n^+.$$

We have just established the following lemma:

Lemma 9.9. *If F has either connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X , and $\perp_{n+1} F(X) \simeq 0$, then*

$$||\perp^* \perp \mathbf{F}(X)|| \simeq \perp \mathbf{F}(X) \wedge_{\Sigma_n} E\Sigma_n^+,$$

where $\perp \mathbf{F}$ denotes the lift to spectra of $\perp F$, as in Corollary 9.6. \square

This ends the establishment of the results required for Proposition 9.3.

9.2.4 Proof Of Proposition 9.3

Recall the statement of the proposition we are to prove:

Proposition 9.10 (Proposition 9.3). *If F has either connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X , and $\perp_{n+1} F(X) \simeq 0$, then $F(X) \simeq P_n^d F(X)$.*

Proof of Proposition 9.3. The proof has four steps: first, settle the trivial case when $n = 0$; second, reduce to the case of a reduced functor, so we can assume $F(0) = 0$. Third, establish the base case $n = 1$. Finally, finish the proof using an induction that involves Proposition 9.11 for lower values of n .

When $n = 0$, the hypothesis of the proposition is that $\perp_1 F(X) \simeq 0$. But $\perp_1 F(X)$ is the fiber of the map $F(X) \rightarrow F(0)$, and this map must be surjective because it has a section induced by $0 \rightarrow X$. Therefore, $\perp_1 F(X) \simeq 0$ means $F(X) \simeq F(0)$, so F is a constant functor. But this is what $P_0^d F(X)$ is as well.

To reduce to the case of a reduced functor, consider the fibration

$$\tilde{F} \rightarrow F \rightarrow F_0,$$

where $F_0(X) = F(0)$ and \tilde{F} is defined to be the homotopy fiber of the map of the map $F(X) \rightarrow F(0)$. Under our hypotheses, Corollary 8.16 shows that P_n^d preserves this fibration, and that it remains surjective on π_0 . If we show that $\tilde{F}(X) \simeq P_n^d \tilde{F}(X)$, then we will have two fibration sequences

$$\begin{array}{ccccc} \tilde{F}(X) & \longrightarrow & F(X) & \longrightarrow & F_0(X) \\ \downarrow & & \downarrow & & \downarrow \\ P_n^d \tilde{F}(X) & \longrightarrow & P_n^d F(X) & \longrightarrow & P_n^d F_0(X) \end{array}$$

with the map on fibers and bases an equivalence. Since $\pi_0 F$ is a group (or 0, which is a group), and the maps to the bases are surjective on π_0 , the Five Lemma applies to show that the map on the total spaces is an equivalence.

For the rest of this proof, we will assume that F is reduced. Given a reduced functor, Corollary 9.7 shows that $F(X) \simeq P_1^d F(X)$, so that establishes the true base case in our induction.

Finally, when $n > 1$ we apply Proposition 9.11 with one smaller n to produce a fibration sequence:

$$||\perp_n^{*+1} F(X)|| \rightarrow F(X) \rightarrow P_{n-1}^d F(X),$$

where the map $F(X) \rightarrow P_{n-1}^d F(X)$ is surjective on π_0 . We now show that the fiber here is equivalent to $D_n^d F(X)$, and then show that this allows us to deduce that the total space must be equivalent to $P_n^d F(X)$.

Using Corollary 9.6, we have

$$||\perp_n^{*+1} F(X)|| \simeq ||\Omega^\infty \perp_n^{*+1} \mathbf{F}(X)||.$$

Using Theorem 6.9, the right hand side is equivalent to

$$\Omega^\infty ||\perp_n^{*+1} \mathbf{F}(X)||.$$

We can then apply Lemma 9.9 to deduce that this is equivalent to

$$\Omega^\infty (\perp_n \mathbf{F}(X) \wedge_{\Sigma_n} E\Sigma_n^+).$$

Corollary 9.6 shows that when $\perp_{n+1} F(X) \simeq 0$, the functor $\perp_n F(X)$ is actually the infinite loop space of a spectrum with Σ_n action, $\perp_n \mathbf{F}(X)$. As in Lemma 9.5, the spectrum $\perp_n \mathbf{F}(X)$ arises from using the structure maps from suspending the left Kan extension in any coordinate, for example:

$$cr_n(LF)_{(X, \dots, X)}(S^0, \dots, S^0) \xrightarrow{\simeq} \Omega cr_n(LF)_{(X, \dots, X)}(S^1, S^0, \dots, S^0).$$

This is exactly the spectrum defined to be the derivative spectrum $\partial^{(n)} LF_X(*)$

(Definition 8.10):

$$\Omega^\infty \left(\partial^{(n)} L F_X(*) \wedge_{\Sigma_n} E \Sigma_n^+ \right).$$

We can identify $E \Sigma_n^+$ with S^0 if we change the strict orbits to homotopy orbits, giving:

$$\Omega^\infty \left(\partial^{(n)} L F_X(*) \wedge_{h \Sigma_n} S^0 \right).$$

Since $S^0 = (S^0)^{\wedge n}$, we can identify this as the form of $D_n^d F(X)$ given in Equation (8.11) in Section 8.2, so we have shown that

$$||\perp_n^{*+1} F(X)|| \simeq D_n^d F(X).$$

It remains to check that the map

$$||\perp_n^{*+1} F(X)|| \rightarrow F(X)$$

actually induces an isomorphism after applying D_n^d . This happens because in order to compute the derivative spectrum, one stabilizes \perp_n , but when \perp_n is applied to the map above, it becomes an equivalence using a standard “extra degeneracy” argument.

In order to apply $D_n^d(-)(X)$, we apply D_n after $L(-)_X$. Evaluation at S^0 would give $D_n^d F(X)$, but we will show the stronger result that actually the natural transformation of functors $D_n L(\epsilon_F)_X$ is an equivalence.

With the aim of applying the “extra degeneracy” argument, we begin by establishing that after applying $L(-)_X$, the map we are considering is actually equivalent to augmentation map $\epsilon_{L F_X}$. This involves verifying that all of the squares in the diagram below commute. A summary of each step follows the diagram.

To aid the reader in understanding the various transformations, we consider F as a trivial simplicial functor (this makes it possible to distinguish between $L||F||$ and $||L F||$).

$$\begin{array}{ccc}
L||\perp_n^{*+1} F(-)||_X & \xrightarrow{L||\epsilon_F||_X} & L||F||_X \\
\downarrow = & & \downarrow = \\
L||\perp_n^{*+1} F_X(-)|| & \xrightarrow{L||\epsilon_{F_X}||} & L||F_X|| \\
\uparrow \simeq & & \uparrow \simeq \\
||L \perp_n^{*+1} F_X(-)|| & \xrightarrow{||L \epsilon_{F_X}||} & ||L F_X|| \\
\downarrow \simeq & & \downarrow = \\
||\perp_n^{*+1} L F_X(-)|| & \xrightarrow{||\epsilon_{L F_X}||} & ||L F_X||
\end{array}$$

The first transformation applied is that by expanding the definition, one can check that $(\perp^k F)_X = \perp^k(F_X)$. The second transformation is the map from $||L(-)|| \rightarrow L||-||$, which is really the commuting of realization in two differ-

ent directions. The third transformation is the commuting of \perp and L , from Lemma 8.17.

Now to compute the coefficient spectra of both sides, we apply \perp_n . Since \perp_n commutes with realizations of functors that satisfy Hypothesis 8.1 or Hypothesis 8.2 (Lemma 9.27), Lemma 9.28 (the “extra degeneracy” argument) shows that the map

$$\perp_n \|\perp_n^{*+1} LF_X\| \xrightarrow{\simeq} \perp_n \|LF_X\|$$

is an equivalence. Stabilizing this (as in Definition 8.10) produces an equivalence on the derivatives, so

$$D_n^d \|\perp_n^{*+1} F\| \xrightarrow{\simeq} D_n^d \|F\|$$

is an equivalence (using Equation (8.11) to compute D_n given the derivative), and in particular,

$$D_n^d \|\perp_n^{*+1} F(X)\| \simeq D_n^d \|F(X)\|$$

via the augmentation map, as desired.

Applying P_n^d to our original fibration gives us a commutative diagram:

$$\begin{array}{ccccc} \|\perp_n^{*+1} F(X)\| & \longrightarrow & F(X) & \longrightarrow & P_{n-1}^d F(X) \\ \downarrow \simeq & & \downarrow & & \downarrow = \\ D_n^d F(X) & \longrightarrow & P_n^d F(X) & \longrightarrow & P_{n-1}^d F(X) \end{array}$$

In particular, this tells us that we may regard the top row as the fibration:

$$D_n^d F(X) \rightarrow F(X) \rightarrow P_{n-1}^d F(X),$$

and recall that this is surjective on π_0 . Alternatively, before evaluation at S^0 , this is:

$$D_n(LF_X) \rightarrow LF_X \rightarrow P_{n-1}(LF_X).$$

The base and the fiber of this fibration are n -excisive, and F on coproducts of X (and hence also the functor LF_X) is either connected (Hypothesis 8.1) or has π_0 a group (Hypothesis 8.2), so Lemma 3.21 shows LF_X is n -excisive. Therefore, $LF_X \simeq P_n(LF_X)$, and then evaluation at S^0 gives $F(X) \simeq P_n^d F(X)$. \square

9.3 Case: $\perp F \not\approx 0$

In this section, the goal is to establish the other side of the “ladder induction” for Theorem 9.1.

Proposition 9.11. *If F has either connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X , and $\perp_{n+1} F(X) \not\approx 0$, then the*

following is a fibration sequence up to homotopy:

$$||\perp_{n+1}^{*+1} F(X)|| \xrightarrow{\epsilon} F(X) \rightarrow P_n^d F(X). \quad (9.12)$$

Furthermore,

$$\pi_0 P_n^d F(X) \cong \text{coker} \left(\pi_0 ||\perp_{n+1}^{*+1} F(X)|| \rightarrow \pi_0 F(X) \right),$$

where coker is the cokernel in the category of groups.

We begin with a definition for the homotopy fiber of the map ϵ .

Definition 9.13 (A_F). Define the functor $A_F(X)$ to be the homotopy fiber in the quasifibration:

$$A_F(X) \rightarrow ||\perp_{n+1}^{*+1} F(X)|| \rightarrow F(X). \quad (9.14)$$

We now outline the proof of this result, essentially as sketched in Section 9.1. We consider the auxiliary diagram:

$$\begin{array}{ccccc} A_F(X) & \longrightarrow & ||\perp_{n+1}^{*+1} F(X)|| & \xrightarrow{\epsilon} & F(X) \\ \downarrow & & \downarrow & & \downarrow \\ P_n^d A_F(X) & \longrightarrow & P_n^d (||\perp_{n+1}^{*+1} F(X)||) & \longrightarrow & P_n^d F(X) \end{array}$$

We show that the bottom row is a quasifibration (Propositions 9.30 and 9.31). We further show that $\perp_{n+1} A_F(X) \simeq 0$ (Lemma 9.29), and hence the case $\perp_{n+1} F \simeq 0$ shows that there is an equivalence of the fibers, so the square on the right is Cartesian. It is not hard to establish that $P_n^d (||\perp_{n+1}^{*+1} F(X)||) \simeq 0$ (Lemma 9.26), so that (9.2) is actually a quasifibration. Especially in Propositions 9.30 and 9.31, attention to path components is needed to let us make the statement about surjectivity on π_0 . Section 9.3.5 assembles all of the ingredients into a proof of the result.

9.3.1 Functors To Groups: $\perp G^{ab} = 0$

This section establishes a technical result that is needed in the proof of Proposition 9.31, where we consider functors to discrete groups.

Let G be a functor from spaces to groups. Generally these functors will arise as lifts of functors from spaces to spaces. For example, π_0 of loops on a space, $F(X) = \pi_0 \Omega X$, lifts to a group-valued functor $G(X)$ by using concatenation of loops for the group operation. In this section, we establish that \perp preserves short exact sequences of groups, and use this to show that the “abelianization” of G has vanishing (n^{th}) cross effect.

Our motivation for following notation comes from the case when the source and target category under consideration are both the category of groups and

the functor G is the identity $G(H) = H$, we have $\perp_2 G(H) = [H * 1, 1 * H]$, and the image of $\perp_2 G(H)$ in $G(H)$ is the first derived subgroup of H . The cokernel of the map $\perp_2 G(H) \rightarrow G(H)$ is the abelianization, H^{ab} . See Section 10.2 for a detailed explanation.

Definition 9.15. Given an $n > 0$ and a functor G to groups, define $G'_n := \text{Im}(\epsilon : \perp_{n+1} G \rightarrow G)$ and $G_n^{ab} := \text{coker}(\epsilon)$. Usually, the n is clear from context, and we will abbreviate these G' and G^{ab} .

There is a short exact (fibration) sequence of groups

$$G'(X) \rightarrow G(X) \rightarrow G^{ab}(X), \quad (9.16)$$

and this sequence is surjective on π_0 (*i.e.*, right exact).

To ease the reader's concern about potentially modding out by a subgroup that is not normal, we note that $G'(X)$ is normal in $G(X)$.

Lemma 9.17. $G'(X)$ is a normal subgroup of $G(X)$.

Proof. $\perp_{n+1} G(X)$ is constructed as the kernel of a map, so it is a normal subgroup of $G(\bigvee^{n+1} X)$. The map $G(\bigvee^{n+1} X) \rightarrow G(X)$ is surjective, so normal subgroups correspond. That is, the normal subgroup $\perp_{n+1} G(X)$ of $G(\bigvee^{n+1} X)$ maps to a normal subgroup $G'(X)$ in $G(X)$. \square

Definition 9.18. Let \perp^{strict} denote the functor identical to \perp , except with the construction made using strict inverse limits or fibers, rather than homotopy inverse limits or homotopy fibers.

Note that there is a natural transformation $\perp^{\text{strict}} \rightarrow \perp$ arising from the canonical map from the strict inverse limit to the homotopy inverse limit.

We use the functor \perp^{strict} in what follows because it is easier to see that a certain functor has $\perp^{\text{strict}} F(X)$ strictly 0 than to show that the simplicial space $||\perp^{*+1} F(X)||$ is contractible.

Lemma 9.19. If F takes values in discrete groups on coproducts of X , then $\perp^{\text{strict}} F(X) \simeq \perp F(X)$.

Proof. The cube defining \perp (and \perp^{strict} ; it is the same cube) has compatible section maps to all structure maps. Since all vertices are discrete, this means that all of the structure maps are fibrations. Taking iterated fibers or homotopy fibers, this implies that the homotopy fiber is equivalent to the strict fiber. \square

Lemma 9.20. If F takes values in discrete groups on coproducts of X , then the image of $\epsilon^{\text{strict}} : \perp^{\text{strict}} F(X) \rightarrow F(X)$ is the same as the image of $\epsilon : \perp F(X) \rightarrow F(X)$.

Proof. The functor $\perp F$ lies between $\perp^{\text{strict}} F$ and the F , so we need to make sure that the image $\perp F \rightarrow F$ is not larger than the image $\perp^{\text{strict}} F \rightarrow F$. From

Lemma 9.19, we know that $\perp^{\text{strict}} F(X) \simeq \perp F(X)$. The space $F(X)$ is discrete, so $F(X) \cong \pi_0 F(X)$. We have the following commutative diagram:

$$\begin{array}{ccccc} \perp^{\text{strict}} F(X) & \longrightarrow & \perp F(X) & \longrightarrow & F(X) \\ \downarrow & & \downarrow & & \downarrow \cong \\ \pi_0 \perp^{\text{strict}} F(X) & \xrightarrow{\cong} & \pi_0 \perp F(X) & \longrightarrow & \pi_0 F(X) \end{array}$$

The bottom row shows that the images of $\pi_0 \perp^{\text{strict}} F(X)$ and $\pi_0 \perp F(X)$ in $\pi_0 F(X)$ coincide. The fact that the right hand vertical map is an isomorphism then implies that the images of $\perp^{\text{strict}} F(X)$ and $\perp F(X)$ in $F(X)$ coincide. \square

Corollary 9.21. *If G takes values in discrete groups on coproducts of X , then the functors $G'(X)$ and $G^{ab}(X)$ of Definition 9.15 can be defined using ϵ or ϵ^{strict} .*

Proof. Lemma 9.20 shows that the images of ϵ and ϵ^{strict} are the same, and the image is all that is used to define G' and G^{ab} . \square

Lemma 9.22. *Suppose that a natural transformation $F \rightarrow G$ is k -connected when evaluated on coproducts of X . Then $\perp F(X) \rightarrow \perp G(X)$ is k -connected.*

Proof. Briefly, this follows because the cubes defining $\perp F(X)$ and $\perp G(X)$ have compatible sections to all structure maps (7.11), so, as in Lemma 7.13, taking the fiber in any direction produces split short exact sequences on homotopy. In this case, a k -connected map on the total space and base of the (quasi-)fibration results in a k -connected map on the fiber. The compatible sections pass to compatible sections on the fibers, so this argument shows that the map on total fibers is k -connected. \square

Corollary 9.23. *Suppose that a natural transformation $F \rightarrow G$ is surjective when evaluated on coproducts of X . Then $\perp^{\text{strict}} F(X) \rightarrow \perp^{\text{strict}} G(X)$ is surjective.*

Proof. When taking strict fibers, an argument almost identical to that in Lemma 9.22 shows that surjectivity is preserved. \square

Lemma 9.24. *Let $A \rightarrow B \rightarrow C$ be a natural short sequence of functors to discrete groups that is a fiber (cofiber) sequence on coproducts of X . Then $\perp^{\text{strict}} A \rightarrow \perp^{\text{strict}} B \rightarrow \perp^{\text{strict}} C$ is a fiber (cofiber) sequence when evaluated at X .*

Proof. The construction of \perp^{strict} involves taking fibers, so it certainly preserves fiber sequences.

A cofiber sequence of discrete groups is a fiber sequence of the underlying sets with the additional property that it is surjective (on π_0). Since \perp^{strict} preserves fiber sequences and Corollary 9.23 shows that \perp^{strict} preserves connectivity (in particular, surjectivity), \perp^{strict} preserves cofiber sequences as well. \square

Lemma 9.25. *If F takes values in discrete groups on coproducts of X , then with G' and G^{ab} as in Definition 9.15, $\perp G^{ab}(X) \simeq 0$.*

Proof. By Lemma 9.19, $\perp G^{ab} \simeq \perp^{\text{strict}} G^{ab}$, so it suffices to show $\perp^{\text{strict}} G^{ab} = 0$. Recall from Corollary 9.21 that we can build G' and G^{ab} using \perp^{strict} instead of \perp .

The map $\epsilon^{\text{strict}} : \perp^{\text{strict}} G \rightarrow G$ factors through $G'(X)$ since the following diagram commutes and $G'(X)$ is the image of ϵ in $G(X)$.

$$\begin{array}{ccc} \perp^{\text{strict}} G(X) & \longrightarrow & \perp G(X) \\ & \searrow \epsilon^{\text{strict}} & \downarrow \epsilon \\ & & G(X) \end{array}$$

That gives us the following factorization of ϵ^{strict} :

$$\begin{array}{ccc} \perp^{\text{strict}} G(X) & & \\ \downarrow & \searrow \epsilon^{\text{strict}} & \\ G'(X) & \longrightarrow & G(X) \end{array}$$

Applying \perp^{strict} to this factorization, we have the factorization:

$$\begin{array}{ccc} \left(\perp^{\text{strict}}\right)^2 G & & \\ \downarrow & \searrow & \\ \perp^{\text{strict}} G' & \longrightarrow & \perp^{\text{strict}} G \end{array}$$

The map $\left(\perp^{\text{strict}}\right)^2 G \rightarrow \perp^{\text{strict}} G$ is surjective, since it has a section map δ (from the diagonal). Therefore, the map $\perp^{\text{strict}} G' \rightarrow \perp^{\text{strict}} G$ must be surjective.

From Lemma 9.24, applying \perp^{strict} to the cofiber sequence (9.16) results in the short exact sequence

$$\perp^{\text{strict}} G' \rightarrow \perp^{\text{strict}} G \rightarrow \perp^{\text{strict}} G^{ab}.$$

We have just shown that the first map is surjective, so the cofiber $\perp^{\text{strict}} G^{ab}$ is zero. \square

9.3.2 If $m < n$, Then $P_m^d || \perp_n^{*+1} F || \simeq 0$

This section establishes the relatively easy fact that for \perp_n , the part of the Goodwillie tower below degree n is trivial.

Lemma 9.26. *Let $R(X_1, \dots, X_n) = ||cr_n(\perp_n^* F)(X_1, \dots, X_n)||$ be a functor of n variables. Define the diagonal of such a functor to be the functor of one*

variable given by $(\text{diag } R)(X) = R(X, \dots, X)$. Then $P_m^d(\text{diag } R)(X) \simeq 0$ for $0 \leq m < n$.

Proof. Goodwillie's Lemma 2.1 [16, Lemma 2.1] shows that if $H(X_1, \dots, X_n)$ is a functor of n variables that is contractible whenever some X_i is contractible (this is called a “multi-reduced” functor), then $P_m(\text{diag } H) \simeq 0$ for $0 \leq m < n$.

Recall that we write L for the left Kan extension of a functor along the full subcategory of spaces generated by coproducts of zero dimensional spheres: $\bigvee^k S^0$. By analogy with the notation $F_X(Y) := F(X \wedge Y)$, let us define $R_X(Y_1, \dots, Y_n) := R(X \wedge Y_1, \dots, X \wedge Y_n)$.

To use Goodwillie's lemma, we need to show that the computation of $P_m^d(\text{diag } R)$ results in computing P_m of the diagonal of a multi-reduced functor. This is an easy computation:

$$\begin{aligned} P_m^d(\text{diag } R)(X) &= P_m L[(\text{diag } R)_X](S^0) \\ &= P_m L[\text{diag}(R_X)](S^0) \\ &= P_m \text{diag}[L^{(n)}(R_X)](S^0), \end{aligned}$$

where $L^{(n)}R$ indicates L applied to each of the n inputs to R separately. It remains to check that $L^{(n)}R_X$ is contractible when any of its inputs is contractible. Since we use the homotopy invariant left Kan extension, if Y is contractible, then $L^{(n)}R_X(Y, \dots) \simeq L^{(n)}R_X(0, \dots)$, and the latter is equivalent to $L^{(n-1)}R_X(0, \dots)$ (removing the L in the first variable), because the Kan extension is equivalent to the original functor on finite sets. Now all n^{th} cross effects have the property that they are contractible if any of their inputs are contractible, so we are done. \square

9.3.3 The Functor A_F Has No $n + 1$ Cross Effect

Having created the functor A_F to be “ F with the cross effect killed”, we now need to establish that $\perp A_F \simeq 0$. The main issue is the commuting of the \perp and the realization.

The essence of the following lemma is that the cubes used to construct $\perp F$ are nice enough that we can compute the cross effects of some particular simplicial functors levelwise.

Lemma 9.27. *Let \perp denote \perp_n for any fixed $n > 0$. If F satisfies Hypothesis 8.1 (connected values) or Hypothesis 8.2 (group values) on coproducts of X , then*

$$\perp ||\perp^{*+1} F(X)|| \simeq ||\perp^{*+2} F(X)||.$$

Proof. If F satisfies Hypothesis 8.1, then $\perp F$ also satisfies Hypothesis 8.1, since \perp preserves the connectivity of the natural transformation from F to the constant zero functor, by Lemma 9.22. Therefore we can compute the fibers in

the \perp -cube levelwise using Waldhausen's Lemma (Lemma 2.10). If F satisfies Hypothesis 8.2, then we will use Theorem 2.12 (Bousfield-Friedlander) to produce the same result. Recall that we use the term “ \perp -cube” to denote the cube whose total (homotopy) fiber is $\perp F$. Since F is a functor to groups, so is $\|\perp^{*+1} F(X)\|$, so each corner of the \perp -cube satisfies the π_* -Kan condition. Furthermore, all of the maps in the \perp -cube have compatible sections (Lemma 7.12), so at each stage of taking iterated fibers all of the structure maps have sections. This gives us surjective maps of simplicial groups, so the induced maps on π_0 are fibrations. These two conditions are enough to apply Theorem 2.12 to compute the fibers levelwise. \square

Lemma 9.28.

$$\|\perp^{*+2} F(X)\| \simeq \perp F(X)$$

Proof. The degeneracy map $\delta : \perp F \rightarrow \perp^2 F$ shows that $\perp F$ is the augmentation of $\|\perp^{*+2} F(X)\|$, so this lemma follows from [26, Exercise 8.4.6, p. 275]. \square

Lemma 9.29. *Let F be a functor satisfying Hypothesis 8.1 or Hypothesis 8.2 on coproducts of X , let \perp denote \perp_n for some n , and let A_F be the functor given in Definition 9.13. Then A_F satisfies $\perp A_F(X) \simeq 0$.*

Proof. Taking cross effects is a homotopy inverse limit construction, and homotopy inverse limits commute, so

$$\begin{aligned} \perp A_F(X) &= \perp \operatorname{hofib} (\|\perp^{*+1} F(X)\| \rightarrow F(X)) \\ &\simeq \operatorname{hofib} (\perp \|\perp^{*+1} F(X)\| \rightarrow \perp F(X)). \end{aligned}$$

Which, by Lemma 9.27, is

$$\simeq \operatorname{hofib} (\|\perp \perp^{*+1} F(X)\| \rightarrow \perp F(X)),$$

and by Lemma 9.28, this is

$$\begin{aligned} &\simeq \operatorname{hofib} (\perp F(X) \rightarrow \perp F(X)) \\ &\simeq 0. \end{aligned}$$

\square

9.3.4 P_n^d Preserves A_F Fibration

This section establishes that P_n^d actually produces a fibration when applied to the fibration defining A_F . The case of F taking values in discrete groups is the most important. Here we actually only show that this is true for F taking values in discrete groups or connected spaces; that is all that is needed to establish the main result that we want.

The results in this section also contain a statement about the map from $F \rightarrow P_n^d F$, because in the case of F taking values in discrete groups, the proof that this map is surjective on π_0 uses the same technical details that the proof that we get a fibration.

Proposition 9.30. *If F satisfies Hypothesis 8.1 (connected values) on coproducts of X , then the following is a quasifibration:*

$$P_n^d A_F(X) \rightarrow P_n^d (||\perp_{n+1}^{*+1} F(X)||) \rightarrow P_n^d F(X),$$

and furthermore the map $F(X) \rightarrow P_n^d F(X)$ is (trivially) surjective on π_0 .

Proof. If F has connected values on coproducts of X , then

$$A_F(X) \rightarrow ||\perp_{n+1}^{*+1} F(X)|| \rightarrow F(X)$$

is a fibration over a connected base. Therefore, by Corollary 8.16, applying P_n^d yields a fibration, so Equation (9.32) is a fibration.

To establish surjectivity of the map $\pi_0 F(X) \rightarrow \pi_0 P_n^d F(X)$, it suffices to show $\pi_0 P_n^d F(X) = 0$. Consider the natural transformation η from the zero functor 0 to F . Since F has connected values on coproducts of X , the map $\eta : 0 \rightarrow F$ is 0-connected on coproducts of X . Applying Theorem 8.9 shows that $0 \simeq P_n^d(0) \rightarrow P_n^d F(X)$ is 0-connected as well. Hence $\pi_0 P_n^d F(X) = 0$. \square

To remind the reader that the functor takes values in discrete groups in the next proposition, we use the letter G (for group) to denote the functor, instead of the usual F .

Proposition 9.31. *If G takes values in discrete groups on coproducts of X (so in particular G satisfies Hypothesis 8.2), then the following is a quasifibration:*

$$P_n^d A_G(X) \rightarrow P_n^d (||\perp_{n+1}^{*+1} G(X)||) \rightarrow P_n^d G(X). \quad (9.32)$$

Proof. Replacing the base G in the definition of A_G (Equation (9.14)) with G' from Definition 9.15, we have the fibration sequence

$$A_G(X) \rightarrow ||\perp_{n+1}^{*+1} G(X)|| \rightarrow G'(X), \quad (9.33)$$

and this sequence is surjective on π_0 . The hypotheses of Corollary 8.16 are satisfied by the sequences in (9.16) and (9.33), so applying P_n^d both are fibration sequences whose maps to the base spaces are surjective on π_0 :

$$P_n^d G'(X) \rightarrow P_n^d G(X) \rightarrow P_n^d G^{ab}(X) \quad (9.34)$$

$$P_n^d(A_G)(X) \rightarrow P_n^d(||\perp_{n+1}^{*+1} G(-)||)(X) \rightarrow P_n^d G'(X). \quad (9.35)$$

The aim now is to show that (9.35) remains a fibration when the base $P_n^d G'(X)$ is replaced by $P_n^d G(X)$. From Lemma 9.25, $\perp_{n+1} G^{ab}(X) \simeq 0$, so

Proposition 9.3 shows that $P_n^d G^{ab}(X) \simeq G^{ab}(X)$, which is a discrete space. Then, using the long exact sequence on homotopy, the fibration in (9.34) gives $P_n^d G'(X) \xrightarrow{\simeq} P_n^d G(X)$ except possibly on π_0 , where the map is injective. This is enough to show that changing the base in (9.35) from $P_n^d G'(X)$ to $P_n^d G(X)$ still yields a fibration. That is, (9.32) is a fibration (but perhaps not surjective on π_0). \square

Proposition 9.36. *If G takes values in discrete groups on coproducts of X (so in particular G satisfies Hypothesis 8.2), then*

$$\pi_0 P_n^d G(X) \cong \operatorname{coker} (\pi_0 ||\perp_{n+1}^{*+1} G(X)|| \rightarrow \pi_0 G(X)),$$

where coker denotes the cokernel in the category of groups.

Proof. As in the preceding Proposition 9.31, we have the following fibration sequence that is surjective on π_0 :

$$P_n^d(A_G)(X) \rightarrow P_n^d(||\perp_{n+1}^{*+1} G(-)||)(X) \rightarrow P_n^d G'(X).$$

Lemma 9.26 shows that the total space in this fibration is contractible, and the map to the base is surjective on π_0 , so $\pi_0 P_n^d G'(X) = 0$.

Also following Proposition 9.31, we have the following diagram in which the horizontal rows are fibrations that are surjective on π_0 :

$$\begin{array}{ccccc} G'(X) & \longrightarrow & G(X) & \longrightarrow & G^{ab}(X) \\ \downarrow & & \downarrow & & \downarrow \simeq \\ P_n^d G'(X) & \longrightarrow & P_n^d G(X) & \longrightarrow & P_n^d G^{ab}(X) \end{array}$$

Since $\pi_0 P_n^d G'(X) = 0$, the long exact sequence for the bottom fibration implies that $\pi_0 P_n^d G(X) \cong \pi_0 P_n^d G^{ab}$. The right hand vertical map is an equivalence, again as noted in the preceding proposition, using Lemma 9.25 and Proposition 9.3. Combining these, we have

$$\begin{aligned} \pi_0 P_n^d G(X) &\cong \pi_0 P_n^d G^{ab}(X) \\ &\cong \pi_0 G^{ab}(X), \end{aligned}$$

so we need to establish that $\pi_0 G^{ab}(X)$ is the cokernel of the group map

$$\pi_0 \epsilon : \pi_0 ||\perp_{n+1}^{*+1} G(X)|| \rightarrow \pi_0 G(X).$$

Both $G^{ab}(X)$ and $G(X)$ are discrete, so it suffices to establish that $G^{ab}(X)$ is the (strict) cokernel of the group map $\epsilon : ||\perp_{n+1}^{*+1} G(X)|| \rightarrow G(X)$. The image of this map is $G'(X)$, by the definition of $G'(X)$ (9.15) and the fact that maps from higher iterates of $\perp_{n+1}^* G(X)$ to $G(X)$ factor through $\perp_{n+1} G(X)$. By

definition (9.15), $G^{ab}(X)$ is the cokernel of the inclusion of $G'(X)$ in $G(X)$, so $G^{ab}(X)$ is the cokernel of the map ϵ , as desired. \square

9.3.5 Proof Of Proposition 9.11

Proposition 9.37 (Proposition 9.11). *If F has either connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X , and $\perp_{n+1} F(X) \not\simeq 0$, then the following is a fibration sequence up to homotopy:*

$$||\perp_{n+1}^{*+1} F(X)|| \xrightarrow{\epsilon} F(X) \rightarrow P_n^d F(X). \quad (9.38)$$

Furthermore,

$$\pi_0 P_n^d F(X) \cong \text{coker}(\pi_0 ||\perp_{n+1}^{*+1} F(X)|| \rightarrow \pi_0 F(X)),$$

where coker is the cokernel in the category of groups.

Proof. First, suppose that $F(X)$ takes either connected values or discrete group values on coproducts of X . Consider the auxiliary diagram created by applying P_n^d to the fibration sequence defining $A_F(X)$:

$$\begin{array}{ccccc} A_F(X) & \longrightarrow & ||\perp_{n+1}^{*+1} F(X)|| & \xrightarrow{\epsilon} & F(X) \\ \downarrow & & \downarrow & & \downarrow \\ P_n^d A_F(X) & \longrightarrow & P_n^d (||\perp_{n+1}^{*+1} F(X)||) & \longrightarrow & P_n^d F(X) \end{array}$$

Proposition 9.30 (in the case of connected values) or Proposition 9.31 (in the case of discrete group values) shows that the bottom row is a quasifibration. Proposition 9.30 (connected values) or Proposition 9.36 (discrete group values) imply that the map $F(X) \rightarrow P_n^d F(X)$ is surjective on π_0 . Lemma 9.29 shows that $\perp_{n+1} A_F(X) \simeq 0$, so that Proposition 9.3 gives $A_F(X) \simeq P_n^d A_F(X)$, and hence the square on the right is homotopy Cartesian. Lemma 9.26 shows that $P_n^d (||\perp_{n+1}^{*+1} F(X)||) \simeq 0$, so this square being Cartesian is equivalent to (9.38) being a quasifibration, as we wanted to establish.

We can reduce the general problem when F satisfies Hypothesis 8.2 to the cases of connected and discrete group values that we have already considered by examining the fibration

$$\widehat{F}(X) \rightarrow F(X) \rightarrow \pi_0 F(X),$$

where $\widehat{F}(X)$ is the component of the basepoint in $F(X)$. This gives rise to the

following square:

$$\begin{array}{ccccc}
||\perp_{n+1}^{*+1} \widehat{F}(X)|| & \longrightarrow & \widehat{F} & \longrightarrow & P_n^d \widehat{F} \\
\downarrow & & \downarrow & & \downarrow \\
||\perp_{n+1}^{*+1} F(X)|| & \longrightarrow & F & \longrightarrow & P_n^d F \\
\downarrow & & \downarrow & & \downarrow \\
||\perp_{n+1}^{*+1} \pi_0 F(X)|| & \longrightarrow & \pi_0 F & \longrightarrow & P_n^d \pi_0 F
\end{array}$$

- The middle column is a fibration by construction, and surjective on π_0 for the same reason.
- The functors F and $\pi_0 F$ are group-valued, and F surjects onto $\pi_0 F$, so Corollary 8.16 shows that the right column is a fibration and surjective on π_0 .
- The left column is a realization of a levelwise fibration. Since F satisfies Hypothesis 8.2, $\perp_{n+1}^{*+1} F(X)$ and $\perp_{n+1}^{*+1} \pi_0 F(X)$ are simplicial groups, and hence satisfy the π_* -Kan condition (2.11).

From Lemma 7.13, we know that $\pi_k \perp_{n+1}^{*+1} F(X) \cong \perp_{n+1}^{*+1} \pi_k F(X)$, so the map

$$\pi_0 \perp_{n+1}^{*+1} F(X) \rightarrow \pi_0 \perp_{n+1}^{*+1} \pi_0 F(X)$$

is an isomorphism of simplicial sets, and hence a fibration. This is the necessary data to apply Theorem 2.12 (Bousfield-Friedlander) to conclude that the realization is a fibration.

The realization of a levelwise 0-connected map is 0-connected, so the left column is also a surjection on π_0 .

- The functor \widehat{F} has connected values, so the top row is a fibration and surjective on π_0 by Proposition 9.30.
- The composition of the maps in the middle row is null homotopic, since the composition factors through $P_n^d (||\perp_{n+1}^{*+1} F(X)||)$, which is contractible by Lemma 9.26.
- The functor $\pi_0 F$ takes values in discrete groups, so the bottom row is a fibration and surjective on π_0 by Proposition 9.31.

We can then use the 3×3 lemma for fibrations to show that the middle row (*i.e.*, (9.32)) is a fibration and surjective on π_0 .

The statement about π_0 is trivial in the connected case; π_0 of every space in the top row is zero (which is trivially a group). This implies that the vertical arrows connecting the second and third rows are π_0 -isomorphisms, so the statement about π_0 follows from Proposition 9.36. \square

Chapter 10

Consequences of the Main Theorem

In this chapter, we explore two of the main consequences of the Main Theorem (9.1). The first is the existence of a spectral sequence that can be used to compute $P_n^d F$, and the second is a way of understanding the n -excisive approximation to the identity functor as a derived functor of the n^{th} derived subgroup functor.

10.1 Spectral Sequence

Bousfield and Friedlander show that under suitable conditions, given a simplicial space X , there is a spectral sequence for calculating the homotopy groups of the realization of a simplicial space $\pi_*||X.||$ from π_*X . We will use their notation: let $\pi_n^v X$ (homotopy in the vertical direction) denote the simplicial set $[k] \mapsto \pi_n X_k$, and let $\pi_m^h \pi_n^v X$ (homotopy in the horizontal direction) denote $\pi_m ||[k] \mapsto \pi_n X_k||$.

Theorem 10.1 ([8, Theorem B.5]). *If X is a simplicial space that satisfies the π_* -Kan condition (2.11), then there is a spectral sequence with $E_{p,q}^2 = \pi_p^h \pi_q^v X$ converging to $\pi_{p+q} ||X.||$.*

We want to use this result to produce a spectral sequence to calculate the homotopy groups of the homotopy fiber of a map $||X.|| \rightarrow Y$.

Corollary 10.2. *Let X be a simplicial space that satisfies the π_* -Kan condition (2.11), and suppose there is a simplicial map to a space Y regarded as a trivial simplicial space. Then there is a spectral sequence converging to $\pi_{s+t} \text{hofib} (||X|| \rightarrow Y)$ whose E^2 term is:*

$$\begin{aligned} E_{s,t}^2 &= \pi_s^h \pi_t^v X & s \geq 1 \\ E_{0,t}^2 &= \ker (\pi_0^h \pi_t^v X \rightarrow \pi_t Y) & s = 0 \\ E_{-1,t}^2 &= \text{coker} (\pi_0^h \pi_t^v X \rightarrow \pi_t Y) & s = -1, t \geq 1 \end{aligned}$$

Proof. As in the proof of Theorem 10.1, let P_t be the Postnikov tower functor with

$$\pi_j P_t Y \cong \begin{cases} \pi_j Y & j \leq t \\ 0 & j > t \end{cases}$$

Let F_t denote the homotopy fiber of the canonical map $P_t \rightarrow P_{t-1}$, so $F_t Y \simeq K(\pi_t Y, t)$ is an Eilenberg-MacLane space. Create the following diagram, letting A_t be the homotopy fiber of

$$||[k] \mapsto P_t X_k|| \rightarrow ||[k] \mapsto P_t Y|| = P_t Y,$$

and $W_t = \text{hofib}(A_t \rightarrow A_{t-1})$:

$$\begin{array}{ccccc} W_t & \longrightarrow & ||F_t X.|| & \longrightarrow & F_t Y \\ \downarrow & & \downarrow & & \downarrow \\ A_t & \longrightarrow & ||P_t X.|| & \longrightarrow & P_t Y \\ \downarrow & & \downarrow & & \downarrow \\ A_{t-1} & \longrightarrow & ||P_{t-1} X.|| & \longrightarrow & P_{t-1} Y \end{array}$$

Note that the middle column is a fibration because P_t preserves the π_* -Kan condition, so Theorem 2.12 applies. Let Z denote the homotopy fiber of the map $||X.|| \rightarrow Y$. The long exact sequence on homotopy, combined with the fact that the natural transformation from the identity to P_t is an isomorphism on homotopy groups in dimensions $\leq t$, shows that $\pi_j A_t \cong \pi_j P_t Z$ for $j < t$. Hence the map $Z \rightarrow A_t$ is at least $(t-1)$ -connected. Since the connectivity of this map increases with t , the spectral sequence derived from the tower of fibrations $\{A_t\}$ converges to the $\pi_* Z$.

To form this spectral sequence, we need to identify $\pi_{s+t} W_t$. Using the long exact sequence from the top row of the diagram, we have:

$$\pi_{s+t} W_t \cong \begin{cases} \pi_{s+t} ||F_t X.|| & s \geq 1 \\ \ker(\pi_t ||F_t X.|| \rightarrow \pi_t Y) & s = 0 \\ \text{coker}(\pi_t ||F_t X.|| \rightarrow \pi_t Y) & s = -1, t \geq 1 \end{cases}$$

The appearance of $s = -1$ occurs because if the map $||X.|| \rightarrow Y$ is not surjective on π_t , the fiber has a homotopy group in one dimension lower. In this case, $t \geq 1$, since failure to be surjective on π_0 is not visible in the fiber. There are no terms with $s \leq -2$ since both $||F_t X.||$ and $F_t Y$ have no homotopy below dimension t . Then, exactly as in [8, p. 123], we can identify $\pi_{s+t} ||F_t X.||$ with $\pi_s^h \pi_t^v X$. This gives us the desired result. \square

We can apply this corollary in the setting of Theorem 9.1 to produce the following result:

Theorem 10.3. *Let F be a homotopy functor from spaces to spaces that takes connected values (Hypothesis 8.1) or group values (Hypothesis 8.2) on coproducts of X . Then there is a spectral sequence beginning with the E^2 page given below and converging to $\pi_{s+t}P_n^dF(X)$.*

$$\begin{aligned} E_{s,t}^2 &= \pi_{s-1}^h \pi_t^v \perp_{n+1}^{*+1} F(X) & s &\geq 2 \\ E_{1,t}^2 &= \ker(\pi_0^h \pi_t^v \perp_{n+1}^{*+1} F(X) \rightarrow \pi_t F(X)) & s &= 1 \\ E_{0,t}^2 &= \operatorname{coker}(\pi_0^h \pi_t^v \perp_{n+1}^{*+1} F(X) \rightarrow \pi_t F(X)) & s &= 0 \end{aligned}$$

Proof. The main theorem (9.1) gives us a quasifibration:

$$||\perp_{n+1}^{*+1} F(X)|| \rightarrow F(X) \rightarrow P_n^d F(X),$$

with $\pi_0 P_n^d$ the group cokernel of the map on π_0 from the fiber to the total space.

The simplicial space $\perp_{n+1}^{*+1} F(X)$ satisfies the π_* -Kan condition (2.11) when $F(X)$ is connected or has group values; that is, under Hypothesis 8.1 or Hypothesis 8.2, so the spectral sequence of Corollary 10.2 can be used and converges to $\pi_* \Omega P_n^d F(X)$. Shifting the index s by one gives a spectral sequence converging to $\pi_{s+t-1} \Omega P_n^d F(X) \cong \pi_{s+t} P_n^d F(X)$, for $s+t \geq 1$. The fact that π_0 is the cokernel of the map as claimed (i.e., $\pi_0 P_n^d F(X) \cong E_{0,0}^2$) is established separately in the main theorem (9.1). \square

We will use this spectral sequence extensively later to understand the functors $P_n^d F$ from computations of the iterated cross effects.

10.2 Lower Central Series

In this section, we explain how the main theorem (9.1) demonstrates a relationship between the n -excisive approximations to the identity functor and the lower central series of a simplicial group. We first recall the related classical results of Curtis.

Recall that if G is a group, the r^{th} group in the lower central series of G is denoted $\Gamma_r G$. The group $\Gamma_r G$ is defined recursively, with $\Gamma_2 G = [G, G]$ being the derived subgroup of G , and $\Gamma_r G = [G, \Gamma_{r-1} G]$ for $r > 2$. When G is a simplicial group, $\Gamma_r G$ is defined to be Γ_r applied levelwise to G .

Theorem 10.4 ([11, Theorem 1.4]). *If G is a free simplicial group that is n -connected, $n \geq 0$, then for $r \geq 2$, the map $G \rightarrow G/\Gamma_r G$ is $\lfloor n + \log_2 r \rfloor$ -connected.*

Actually, because G is a free group, the map is an isomorphism on homotopy in dimension $\lfloor n + \log_2 r \rfloor$ as well, but this is not so important. Another way of stating this theorem is that the cofibration (=fibration) sequence of (simplicial) groups

$$\Gamma_r G \rightarrow G \rightarrow G/\Gamma_r G \tag{10.5}$$

is surjective on π_0 and the connectivity of the fiber is at least $\lfloor n + \log_2 r \rfloor$.

The main consequence of this theorem is that Curtis is able to apply it to Kan's loop group functor $G(X)$, which is a simplicial group that in dimension k is the reduced free group on the $(k+1)$ -simplices of X (reduced meaning that the generator corresponding to the basepoint is identified with the identity). Let X be a simply connected simplicial set, and let $G = G(X)$ be the free simplicial group resulting from applying Kan's loop group functor. The theorem then shows that the lower central series filtration "converges", in the sense that $G \simeq \lim_r G/\Gamma_r G$ because the spaces are connected and the connectivity of the fiber of the map grows (slowly) to infinity. This in turn means that ΩX can be analyzed by looking at the quotients in the lower central series.

To analyze the functor $G(X)$ in our setting, let $G(X) = |G(\text{Sing}(X))|$, where $\text{Sing}(X)$ is the standard singular simplicial set of X with $\text{Sing}(X)_k = \text{Hom}(\Delta^k, X)$. For convenience, use the notation $X_k = \text{Sing}(X)_k$. To compute $\perp_n G(X)$, we need to compute the total homotopy fiber of a cube whose vertices are $G(\bigvee^k X)$, for $0 \leq k \leq n$. In general, the cube for any $\perp_r G$ has compatible sections to every structure map, and all of the vertices are simplicial groups (and hence satisfy the π_* -Kan condition (2.11)), so by Theorem 2.12 (Bousfield-Friedlander), we can compute the homotopy fiber by taking fibers of the simplicial sets (groups) levelwise. Furthermore, since everything is fibrant, we can use strict fibers rather than homotopy fibers. When working with groups, the "strict fiber" of a map is commonly called the kernel, so we use that term henceforth.

In dimension m , the simplicial group $G(\bigvee^k X)_m$ is the reduced free group generated by $\bigvee^k X_{m+1}$. The reduced free group functor distributes over coproducts, so this is the free product of k copies of the free group on X_{m+1} .

The second cross effect $\perp_2 G(X)$ is the (realization of the levelwise) kernel of the map $G(X \vee X) \rightarrow G(X) \times G(X)$, which by the preceding paragraph is the same as the kernel of the map $p : G(X) * G(X) \rightarrow G(X) \times G(X)$. To distinguish between the two copies of $G(X)$, we will use $g_i \in G(X) * 1$ and $h_j \in 1 * G(X)$.

Define $\Gamma_2^{\text{ext}} G(X)$ to be the normal closure of the subgroup $[G(X) * 1, 1 * G(X)]$ of $G(X) * G(X)$ generated by commutators of the form $[g_i, h_j]$. We will show that $\Gamma_2^{\text{ext}} G(X)$ is the kernel of the map p , which is $\perp_2 G(X)$. An element $g_1 h_1 g_2 h_2 \cdots g_w h_w \in G(X) * G(X)$ is in the kernel of the map p if the products $g_1 \cdots g_w = e$ and $h_1 \cdots h_w = e$. All commutators that generate $\Gamma_2^{\text{ext}} G(X)$ are of this form, and the kernel of a map is normal, so $\Gamma_2^{\text{ext}} G(X)$ is contained in $\perp_2 G(X)$. Now, commuting the g and h elements to place all of the g 's adjacent and all of the h 's adjacent produces the product of $(g_1 \cdots g_w)(h_1 \cdots h_w) = e$ with a bunch of commutators of g_i and h_j , including higher iterated commutators. We claim that these higher iterated commutators are contained in $\Gamma_2^{\text{ext}} G(X)$. Let $y \in \Gamma_2^{\text{ext}} G(X)$ and let $g \in G * G$: the element $[y, g] = y^{-1} g^{-1} y g$ is then in $\Gamma_2^{\text{ext}} G(X)$ because it is normal. This shows that the kernel of the map p is also contained in the subgroup $\Gamma_2^{\text{ext}} G(X)$, so $\perp_2 G(X) = \Gamma_2^{\text{ext}} G(X)$.

We use the notation $\Gamma_2^{\text{ext}}G(X)$ to denote a group in the “exterior lower central series” since its image under the fold map ϵ is the standard subgroup $\Gamma_2G(X)$ in the lower central series of $G(X)$. The higher cross effects are produced by iterating the second cross effect (7.9), so the same analysis shows that $\perp_r G(X)$ is the r^{th} “exterior” derived subgroup $\Gamma_r^{\text{ext}}G(X)$.

We can now interpret the fibration from Theorem 9.1,

$$||\perp_{r+1}^{*+1} F(X)|| \rightarrow F(X) \rightarrow P_r^d F(X),$$

in the case when the functor $F(X)$ is our functor $G(X)$, which is essentially loops on a space. The functor G requires $X_0 = *$ (so X is connected), and if X is connected, then the functor $G(X \wedge -) \simeq \Omega(X \wedge -)$ commutes with realizations (using Lemma 2.10). Lemma 11.1 then implies that $P_r^d G(X) \simeq P_r G(X)$. This lets us translate the statement of the theorem to the fibration:

$$\left| (\Gamma_{r+1}^{\text{ext}})^{*+1} G(X) \right| \rightarrow G(X) \rightarrow P_r G(X).$$

One way to produce a direct comparison of this spectral sequence with Curtis’s result is to realize that in the setting of bisimplicial groups, it becomes somewhat easier to understand the spectral sequence we are using. Recall that a simplicial group G may be transformed into a nonabelian chain complex NG by the process of normalization (for details, see [26, §8.3, pp. 264–266]). To produce the normalized complex, one takes the intersection of the kernels of all but the last face map d_0 , and uses d_0 for the differential: $(NG)_n = \bigcap_{i>0} \ker d_i$ and $d_n : (NG)_n \rightarrow (NG)_{n-1}$ is the same d_n from the simplicial group G . The homology of the resulting chain complex $H_*(NG)$ is isomorphic to the homotopy of the original simplicial group, $\pi_* G$. Using this process, we see that the spectral sequence of Corollary 10.2 arises from taking the homology of the bicomplex of nonabelian groups:

$$G(X) \leftarrow \Gamma_{r+1}^{\text{ext}} G(X) \leftarrow N(\Gamma_{r+1}^{\text{ext}})^2 G(X) \leftarrow \dots$$

This bicomplex maps to its “good (horizontal) truncation”:

$$G(X) \xleftarrow{\epsilon} \text{Im}(\epsilon) \leftarrow 0 \leftarrow \dots,$$

where $\text{Im}(\epsilon) \cong \Gamma_{r+1} G(X)$, as noted above. Now taking homology horizontally first, then vertically, causes the spectral sequence to collapse to $\pi_*(G(X)/\Gamma_{r+1} G(X))$, which is the approximation Curtis uses. Taking homology vertically first produces:

$$\pi_* G(X) \leftarrow \pi_* \Gamma_{r+1} G(X) \leftarrow 0;$$

then when we take homology horizontally, we can identify the resulting E^2 page as a quotient the E^2 page from Theorem 10.2, with an isomorphism in the first column and a surjection in the second column. This shows that the approxima-

tion Curtis uses can be regarded as the good truncation of an approximation arising from Goodwillie's calculus. The good truncation we used produces H_0 , so in this sense, Curtis's approximation $G/\Gamma_{r+1}G$ is H_0 of the approximation used by Goodwillie calculus. Looking at this another way, the Goodwillie tower can be thought of as the derived functor of the quotient by the lower central series. From the Blakers-Massey Theorem (3.4), the functor $G(X) \simeq \Omega(X)$ satisfies the stable excision condition $E_n(n-1)$. This implies that for X simply connected, the map $G(X) \rightarrow P_r G(X)$ is $(r-1)$ -connected. Comparing this with Curtis's result, we see that the higher columns in the spectral sequence make a tremendous difference; they raise the connectivity from roughly $\log_2 r$ up to roughly r .

Chapter 11

Different Theories Of Calculus

In this chapter, we present a way of combining Goodwillie’s excisive calculus and the additive calculus, and show there is a filtration of distinct “theories” beginning with the additive calculus and ending in the excisive calculus. The additive calculus is in many cases easier to work with than the excisive calculus, since it involves no suspensions. In one respect, this filtration shows how many suspensions are necessary before the additive and excisive approximations become the same.

The building block for the excisive calculus is the T_n functor that increases n -excisiveness. For additive calculus (recall Section 4.3), the “building block” appears to be $T_n(L_0F_X)$; that is, T_n applied to the left Kan extension of F along coproducts of X . In the case $n = 1$, we have $T_1F(X) = \Omega F(S^1 \wedge X)$ and $T_1(L_0F_X)(S^0) = \Omega||F(S^1 \wedge X)||$. However, if the latter functor is iterated, it produces $\Omega||\Omega||F(S^1 \wedge S^1 \wedge X)|||$, which need not have a good behavior with respect to coproducts. To produce the desired additive approximation, one must apply all of the iterations of T_n to the same left Kan extension; *i.e.*, $T_n^k(L_0F_X)(S^0)$.

This limits the number of serious candidates for interpolating between additive and excisive calculus to just a few. Beginning with the sequence of natural transformations

$$\text{Id} \rightarrow T_n \rightarrow T_n^2 \rightarrow \cdots \rightarrow P_n,$$

we can either apply P_n^d to this sequence, or apply each functor in the sequence to P_n^d . Applying P_n^d to the stabilization sequence is a good approach because it isolates complicated behavior that spreads across dimensions. The stabilization of P_n^d (that is, applying this sequence of functors to P_n^d) is also interesting, and we will say some things about it. Actually, the two are very closely related; for functors whose target category is spectra, they are equivalent. However, for functors whose target category is spaces, they are not.

11.1 Background And Basic Facts

Lemma 11.1. *If F commutes with realizations and satisfies the limit axiom, then $P_n^d F(X) \simeq P_n F(X)$.*

Proof. We begin with $P_n^d F(X) = P_n(L_0 F_X)(S^0)$. Now if F commutes with realizations, then $L_0 F_X \simeq F_X$ because they agree on finite sets, and hence all discrete sets by the limit axiom. This means that they agree levelwise on all simplicial sets, and hence also after realization (Lemma 2.1). The right hand side is therefore equivalent to $P_n F(X)$. \square

Lemma 11.2. *If $\pi_0 F$ is additive and $F(X)$ is connected or a grouplike H -space, then $P_1^d(\Omega F(X)) \simeq \Omega P_1^d F(X)$.*

Proof. Under these conditions, the spectral sequence of Theorem 10.3 converges. There is a map $P_1^d(\Omega F(X)) \rightarrow \Omega P_1^d F(X)$ comparing the two functors. Comparing the spectral sequence for $P_1^d(\Omega F(X))$ and $P_1^d F(X)$, we see that they are the same except for a dimension shift and the difference in π_0 that is lost when Ω is applied. \square

Definition 11.3. For convenience, let us define $P_n^{(a)}$ to denote the functor $P_n^d T_n^a$.

Proposition 11.4. *Let F be a reduced r -analytic functor from spaces to spaces satisfying the limit axiom (5.1), with universal analyticity constant c as in Theorem 6.19. Let $M = \max(r, -c)$. Then:*

1. $P_n^d F(X) \simeq P_n F(X)$ if X is at least M -connected.
2. $P_n^{(m)} F \simeq P_n F$ for all for all $m > M$; and

Proof. Theorem 6.19 shows that F commutes with realizations of simplicial M -connected spaces. Lemma 11.1 then shows that $P_n^d F(X) \simeq P_n F(X)$ for X at least M -connected. The space $T_n F(X)$ is computed by evaluating a homotopy inverse limit of a diagram involving suspensions of X . If X is k -connected, then $T_n F(X)$ depends only on F evaluated on $(k+1)$ -connected spaces. Hence $T_n^m F(X)$ depends only on F evaluated on $(k+m)$ -connected spaces. The minimum connectivity of a pointed space is -1 , so $T_n^m F$ depends only on F on $(m-1)$ -connected spaces, and hence under the condition that $m > M$, it commutes with realizations. Then the preceding paragraph shows that $P_n^{(m)} F = P_n^d T_n^m F \simeq P_n T_n^m F \simeq P_n F$, as desired. \square

11.2 The Hilton-Milnor Theorem And Whitehead Products

In order to make computations, we will need to know a bit about Whitehead products and make use of the Hilton-Milnor theorem.

Given $\alpha \in \pi_n X$ and $\beta \in \pi_m X$, there is a product $[\alpha, \beta]$ called the Whitehead product that behaves much like a Lie bracket. For details, see [27, §IX.7].

To define the Whitehead product, consider α as a map of pairs $(D^n, \partial D^n) \rightarrow (X, *)$, and β similarly. If X were an H -space, we could define a map on the product $D^n \times D^m$ by $(x, y) \mapsto \mu(\alpha(x), \beta(y))$. Since that may not be the case, we can consider the map from $S^{n+m-1} \cong \partial(D^n \times D^m) = (\partial D^n) \times D^m \cup D^n \times (\partial D^m)$ given by α on D^n or β on D^m . This definition makes sense since $\alpha|_{\partial D^n} = *$, and similarly β , so either one or the other is $*$ for every point in the domain. This defines a map $[\alpha, \beta] : S^{n+m-1} \rightarrow X$. This element of $\pi_{n+m-1} X$ is called the Whitehead product of α and β .

The main fact about Whitehead products that we will make use of is that they are graded commutative.

Lemma 11.5. ([27, §X.7.5, p. 474]) *If $\alpha \in \pi_n X$ and $\beta \in \pi_m X$, then $[\beta, \alpha] = (-1)^{nm}[\alpha, \beta]$.*

For our purposes, very little of the full strength of the Hilton-Milnor theorem will be needed, so we will give as few details as possible.

Theorem 11.6 (Hilton-Milnor). ([27, §XI.6, Theorem 6.6]) *The space $\bigvee^k S^n$, for $n > 1$, has the same homotopy type as the weak product $\prod S^{w_j}$, where w_j is a sequence of integers beginning with $w_j = n$ for $j = 1, \dots, k$, then increasing in steps of $n - 1$. The equivalence sends S^{w_j} to a certain iterated Whitehead product, with the weight m Whitehead product corresponding to a sphere of dimension $S^{n+(m-1)(n-1)}$.*

Lemma 11.7. *The functor $\pi_m(S^n \wedge -)$ is degree $k = \lfloor (m-1)/(n-1) \rfloor$.*

Proof. The Hilton-Milnor Theorem implies that the homotopy π_m of $\bigvee S^n$ is determined by the basic products of weight at most k . Verification involves checking that the a product of weight $\leq k$ corresponds to a sphere of dimension $k(n-1) + 1 \leq m$, and that a product of weight $k+1$ corresponds to a sphere of dimension $(k+1)(n-1) + 1 = k(n-1) + n > m$. \square

Example 11.8. In particular, Lemma 11.7 shows that $\pi_{2n-1}(S^n \wedge -)$ is the lowest homotopy group of $(S^n \wedge -)$ that is degree 2. Computing $\pi_3(S^2 \vee S^2)$, for example, we find two copies of $\pi_3(S^2) \cong \mathbb{Z}$ and one copy of $\pi_3 S^3 \cong \mathbb{Z}$. Therefore, $\perp_2 \pi_3(S^2 \wedge -) \cong \mathbb{Z}$. Actually, for every n , we have the same: $\perp_2 \pi_{2n-1}(S^n \wedge -) \cong \mathbb{Z}$, since the first new sphere in $S^n \vee S^n$ corresponds to the Whitehead product $[i_1, i_2]$ of the two inclusions $S^n \rightarrow S^n \vee S^n$, and is a copy of S^{2n-1} . We will use this soon in our computations.

11.3 Examples Showing Theories Are Distinct

Notice that Lemma 11.1 shows that under at least some circumstances, $P_n^d F$ and $P_n F$ coincide, so we need to provide some evidence that they can differ.

Example 11.9. One example that is both trivial and fundamental is the functor $F(X) = K(H_2(X), 2)$, as in Example 4.11. This is a functor whose values are always simply connected, but $P_1 F(X) = 0$ (in fact, the whole excisive Taylor tower is zero). We can compute $P_1^d F(S^2) \cong K(\mathbb{Z}, 2)$, since $H_2(\bigvee^k S^2) \cong \bigoplus^k \mathbb{Z}$. This shows that P_n^d and P_n can be very different, and also illustrates that the connectivity of the values of F have nothing to do with that difference. As an aside, the radius of convergence of F is 2; that is, F is equivalent to the inverse limit of its Taylor tower on 2-connected spaces (because both are contractible there).

While important, the preceding example is not such a satisfying way of demonstrating a difference between additive and excisive. Notice, though, that with slight modifications, it does produce one family of examples for which all of the $P_1^{(a)}$ functors are distinct. For the family of functors $F_b(X) = K(H_b(X), b)$, we can compute that $\pi_b P_n^{(a)} F_b(X) \cong H_{b-a} X$. In particular, for a given X and large enough a , this is zero, whereas when a is small relative to the dimension of X , there are many spaces X for which it is not zero (e.g., $X = \bigvee_{k=0}^{k=a} S^k$).

Using Theorem 10.3, we will show that there is another way in which the $P_n^{(a)}$ functors can be different. To make the computations tractable, we will work only with $n = 1$.

Example 11.10. Fix an a and consider the functor $F(X) = \Omega^{3a}(S^a \wedge X)$. We will establish that $P_1^{(a)} F \xrightarrow{\not\sim} P_1^{(a+1)} F$ by evaluating both at S^0 .

First, we will show that $P_1^{(a+1)} F \simeq P_1 F$.

Lemma 11.11. $P_1^{(a+1)} F(S^0) \simeq P_1 F(S^0)$

Proof. We have:

$$T^{a+1} F = \Omega^{4a+1}(S^{2a+1} \wedge -),$$

and $\pi_j(S^{2a+1} \wedge -)$ is linear for $j \leq 4a+1$ (by Lemma 11.7), so by Lemma 11.2,

$$\begin{aligned} P_1^{(a+1)} F &= P_1^d \Omega^{4a+1}(S^{2a+1} \wedge -) \\ &\simeq \Omega^{4a+1} P_1^d(S^{2a+1} \wedge -), \end{aligned}$$

and since $(S^{2a+1} \wedge -)$ commutes with realizations, by Lemma 11.1 this is

$$\begin{aligned} &\simeq \Omega^{4a+1} P_1(S^{2a+1} \wedge -) \\ &\simeq P_1 F, \end{aligned}$$

where the last equivalence follows because $P_1 F$ is 1-excisive, so $\Omega^{a+1} P_1 F(S^{a+1} \wedge X) \simeq P_1 F(X)$. \square

Next, we will show that $P_1^{(a)} F$ is not equivalent to $P_1 F$. We will do this by mapping to another functor that is equivalent to $P_1 F$, for exactly the same reason as the functor in Lemma 11.11.

The functor $T_1^a F$ is $\Omega^{4a}(S^{2a} \wedge -)$, which for our purposes we will view as ΩG , for $G(-) = \Omega^{4a-1}(S^{2a} \wedge -)$. Using Lemma 11.7, we see that G has π_0 quadratic. In fact, putting $v = 2a$, we see that this functor is the functor $\pi_{2v-1}(S^v \wedge -)$ from Example 11.8, so π_0 of $\perp_2 G$ is actually \mathbb{Z} . Our assertion is that the map $P_1^d \Omega G \rightarrow \Omega P_1^d G$ is not even 1-connected, whereas the latter functor is equivalent to $P_1 F$. To show that this map is not 1-connected, we will use the spectral sequence of Theorem 10.3. For computing with that spectral sequence, it will be helpful to note that cross effects commute with taking homotopy groups (Lemma 7.13); we will use this fact without further comment.

Recall from Lemma 9.9 that when $\perp_{n+1} F \simeq 0$, the complex $\perp_n^{*+1} \mathbf{F}$ has the realization $(\perp_n \mathbf{F})_{h\Sigma_n}$, and homotopy orbits can also be expressed as the group homology of Σ_n . In our case, $n = 2$, and the complex is, as usual for $H_*(\Sigma_2; -)$, one with only one nondegenerate cell in each dimension. By coincidence, we actually have the complex for $H_*(\Sigma_2; \mathbb{Z})$, since $\pi_0 \perp_2 \pi_{2v-1}(S^v \wedge -) \cong \mathbb{Z}$, but this would not be the case if, for example, we increased the dimension of the sphere involved by one.

The bottom row of the spectral sequence of Theorem 10.3 arises from the augmented complex $F \leftarrow \perp_n^{*+1} F$, so $E_{0,0}$ is the group $\pi_{2v-1} S^v$. We will not need to know anything about it, just that the complex is still exact with the augmentation map (which it is), so the only differences between the (shifted) $H_{*-1}(\Sigma_2; \mathbb{Z})$ and $E_{*,0}^2$ are in dimension 0 (obviously) and 1.

In the particular case of our functor, we have a good amount of information about the bottom row. On the E^1 page, we do not know $\pi_{2v-1} S^v$, but we have v even, so we know by Serre's work that it contains an infinite cyclic factor (*e.g.*, [7, Theorem 18.22, p. 254]). The rest of the row consists of factors of \mathbb{Z} . As usual when computing the homology of a cyclic group, there is only one non-degenerate copy of \mathbb{Z} in each dimension. Furthermore, the differentials $d_{i,0}^1$ for $i > 1$ are either multiplication by 2 or 0, as we will now explain. Each copy of \mathbb{Z} corresponds to a Whitehead product of two of the inclusion maps $S^v \rightarrow \bigvee S^v$, and we have v even, so the graded commutativity of the Whitehead product means that $[\alpha, \beta] = [\beta, \alpha]$. Specifically, $d_{2,0}^1 = 0$ since the non-degenerate copy of \mathbf{Z} is represented by the product $[i_0 i_1, i_1 i_0]$, which is sent under the differential $d = \partial_0 - \partial_1$ to

$$[i_1, i_0] - [i_0, i_1] = [i_1, i_0] - [i_1, i_0] = 0.$$

Similarly, $d_{3,0}^1 = 2$.

The group $E_{0,0}^1$ contains an infinite cyclic factor as mentioned, but the spectral sequence converges to stable homotopy, so $E_{0,0}^\infty = \pi_{2v-1}^S S^v$ is known to be torsion. Hence the differential $d_{1,0}^1$ must be injective.

This gives us enough information about the spectral sequence to determine that the map $P_1^d \Omega G \rightarrow \Omega P_1^d G$ is not 1-connected. In low dimensions, the E^2

page of the spectral sequence for $P_1^d G$ is the following:

$$\begin{array}{ccccc}
 & & E_{0,2}^2 & & \\
 & \swarrow d_{3,0}^3 & & \searrow & \\
 & E_{1,1}^2 & & & \\
 \swarrow d_{2,0}^2 & & \swarrow d_{3,0}^2 & & \\
 E_{0,1}^2 & & 0 & & \mathbb{Z}/2 \\
 \swarrow d_{2,0}^2 & & & & \\
 E_{0,0}^\infty & & & &
 \end{array}$$

If $d_{2,0}^2$ is nonzero, then the spectral sequence for $P_1^d \Omega G$ (which is just a shifted version of the one for $P_1^d G$) converges to $\pi_0 P_1^d \Omega G = E_{0,1}^2$, whereas $\pi_0 \Omega P_1^d G = E_{0,1}^\infty$.

Now consider the case in which $d_{2,0}^2$ is zero. The group $E_{3,0}^2 = 0$, and hence supports no differentials, so the spectral sequence for $P_1^d \Omega G$ produces the same terms corresponding to $E_{0,2}^2$ and $E_{1,1}^2$ as that for $P_1^d G$ (these groups live to E^∞ because they are not the target of any more differentials). However, $\pi_1 \Omega P_1^d G$ is an extension of $E_{2,0}^2 \cong \mathbb{Z}/2$ by these groups, whereas $\pi_1 P_1^d \Omega G$ has no such factor, so they differ in the abutment.

This example actually shows the difference between the two alternative constructions mentioned at the start of the chapter, since P_n commutes with Ω always, and P_n^d was just shown not to.

Lemma 11.12. *The functor $P_n^d F$ need not be equivalent to either $T_n P_n^d F$ or $P_n^d T_n F$. \square*

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Vita

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